Version 2 for Analysis

Prof. Dr. Hassane Bouzahir

Ibn Zohr University

October 31, 2020

Consequences of Least Upper Bound property of $\mathbb{R}$.

Theorem

($\mathbb{Q}$ is dense in $\mathbb{R}$.) If $x < y \in \mathbb{R}$, then $\exists q \in \mathbb{Q}$ such that $x < q < y$.

Proof.

$y - x > 0$. Thus $\exists n \in \mathbb{N}$ such that $n(y - x) > 1$ by Archimedean property. Also by the Arch. property, $\{nx\}$ is not bounded $(x \neq 0)$. Choose m such that $m - 1 \leq nx < m$, where $m \in \mathbb{N}$. Then $x < \frac{m}{n} < y$ because $ny > nx + 1 \geq m$. □

Properties of Sup

- γ is an upper bound of $A \iff \sup A \leq \gamma$.
- If $\forall a \in A, a \leq \gamma$, then $\sup A \leq \gamma$.
- If $\forall a \in A, a < \gamma$, then $\sup A \leq \gamma$.
- If $\gamma < \sup A$, then $\exists a \in A$ such that $\gamma < a \leq \sup A$.
- If $A \subseteq B$, then $\sup A \leq \sup B$. *Pf*: Let $a \in A$. Then $a \in B$. Thus $a \leq \sup B$. Thus $\sup B$ is an upper bound of A. Thus we say $\sup A \leq \sup B$. $\square$

Definition

Let $n \in \mathbb{N}$. Real Euclidean Space of Dimension n si the set of all n-tuples $\mathbb{R}^n = (x_1, \ldots, x_n), x_i \in \mathbb{R}$. Addition is component-wise, with $\vec{0}$ as the additive identity. However, there is no multiplication that satisfies the axioms of a field. (Dot product returns a scalar, cross product is specific to $\mathbb{R}^3$ and component-wise multiplication doesn't always have inverse. Consider $\vec{x} = (1, 2, 3, 0, \ldots, n)$.)

Remark: $\mathbb{R}^n$ is a vector space, not a field.
$\mathbb{R}^n$ has an inner product:

$$\vec{x} \cdot \vec{y} = \sum_{i=1}^{n} x_i y_i$$

and norm

$$||\vec{x}|| = \sqrt{\vec{x} \cdot \vec{x}}.$$

Complex Numbers

Remark: $\mathbb{R}^n$ can be given structure of a field. Consider

$$\begin{aligned} (+) \quad & (a, b) + (c, d) = (a + c, b + d) \\ (\cdot) \quad & (a, b) \cdot (c, d) = (ac - bd, ad + bc). \end{aligned}$$

With this structure $(\mathbb{R}^2, +, \cdot)$ is called $\mathbb{C}$.

Theorem

$\mathbb{C}$ is a field.

Proof.

Check the axioms. □

Remark: $(0, 0)$ is the additive identity. $(1, 0)$ is the multiplicative identity.

Remark:
$(a,0) \leftrightarrow a \in \mathbb{R}$. $(0,1)$ is called i. Note

$$(0,1)\cdot(0,1) = (-1,0).$$

We can write $(a,b) \leftrightarrow a + bi$.

Definition

The length or modulus of z is $|z| = \sqrt{z\bar{z}}$.

omplex numbers have the following properties:

- $\overline{z+w} = \bar{z} + \bar{w}$.
- $\overline{zw} = \bar{z}\cdot\bar{w}$
- $|z| = |\bar{z}|$
- $|zw| = |z||w|$

Theorem

(Triangle Inequality) $|z+w| \leq |z| + |w|$.

Proof.

$$
\begin{aligned}
|z+w|^2 &= (z+w)(\bar{z}+\bar{w}) \\
&= z\bar{z} + w\bar{w} + zRe(z\bar{w}) \\
&\leq |z|^2 + |w|^2 + 2|z\bar{w}|, \\
&\text{because modulus includes imaginary part} \\
&= (|z|+|w|)^2
\end{aligned}
$$

□

Notation. Henceforth, $\vec{a}, \vec{b} \in \mathbb{C}^n$ will be simply denoted a, b.

Definition

$\mathbb{C}^n=(z_1,\ldots,z_n), z_i \in \mathbb{C}$. *The inner product of* a, b *is given by*

$$\langle a, b\rangle = \sum_{j=1}^{n} a_j\bar{b}_j \geq 0.$$

Theorem

(Cauchy Swarz). Let $\bar{a}, \bar{b} \in \mathbb{C}^n$. *Then*

$$|\langle \bar{a}, \bar{b}\rangle|^2 \leq \langle \bar{a}, \bar{a}\rangle \cdot \langle \bar{b}, \bar{b}\rangle$$

$$i.e., \quad |\langle a, b\rangle| \leq \|a\|\|b\|$$

$$i.e., \quad |\sum a_j b_j|^2 \leq \sum |a_j|^2 \sum |b_j|^2$$

Proof

Let $\lambda \in \mathbb{R}$. Consider

$$\langle a, \lambda b\rangle = \sum a_j \lambda \bar{b}_j = \lambda \sum a \bar{b}_j = \lambda \langle a, b\rangle.$$

Also note that $\|\lambda b\| = \lambda \|b\|$. Thus

$$\langle a, b\rangle \leq \|a\| \cdot \|b\| \iff \langle a, b\rangle \leq \|a\| \|\lambda b\|, \quad \lambda \leq 0.$$

This allows us to say WLOG, $\|b\| = 1$. Now we will decompose a into its components parallel and perpendicular to b.

$$proj_b a = \frac{\langle a, b\rangle}{\|b\|} b = \langle a, b\rangle b.$$

Let $proj_{b^+} a = c$. (c is a vector but here we'll leave off the arrow.)

We then compute

$$\begin{aligned}\|a\|^2 &= \langle a, a\rangle \\ &= \langle \langle a, b\rangle b + c, \langle a, b\rangle b + c\rangle \\ &= |\langle a, b\rangle|^2 \|b\|^2 + \|c\|^2 \\ &= |\langle a, b\rangle|^2 + \|c\|^2 \\ &\geq |\langle a, b\rangle|^2,\end{aligned}$$

since $\|c\|^2$ is nonnegative. Furthermore, since $\|b\| = 1$, we can conclude $\|a\|\|b\| \geq |\langle a, b\rangle|$, as desired.

Definition

Let $f : A \to B$ be a map of sets.

- *For any subset $E \subset A$, the image of E under f is $f(E) \subseteq B$, i.e. $f(E) = \{f(e) \in B / e \in E\}$*
- *The range of f is $F(A)$.*
- *The domain of f is A, The codomain of f is B and f is onto (surjective) if $f(A) = B$.*
- *For $C \subseteq B$, the inverse image of C is $f^{-1}(C) = \{a \in A / f(a) \in C\}$.*
- *f is one-to-one (injective) if for each element $b \in B$, $f^{-1}(b)$ has at most one element.*
- *f is bijective if f is injective and surjective.*
- *A and B are in $1-1$ correspondence, or have the same cardinality, or, also equivalently, the sets are called bijective or equivalent, if $\exists$ a bijection $f : A \to B$.*

Note 1: f is injective $\iff$ $f(a_1) = f(a_2)$ implies $a_1 = a_2$.//
Remark 1: Bijective sets are denoted $A \longleftrightarrow B$ or $A \sim B$. The latter notation is justified because this relation is an equivalence relation.

Definition

For $n \in \mathbb{N}$, let

$$[n] = 1, 2, 3 \ldots, n.$$

Note the book uses j_n equivalently. Let A be a set.

- A is finite if $\exists$ a bijection, $A \longleftrightarrow [n]$ for some $n \in \mathbb{N}$.
- A is infinite if it is not finite.
- A is countable if $A \longleftrightarrow \mathbb{N}$.
- A is at most countable if A is finite or countable.
- A is uncountable if A is not finite and not countable.

Definition

Let A be a set. A sequence in A is a function (i.e., map of sets) $f : \mathbb{N} \to A$. We write $f(n) = a_n$. and $\{a_n\}$ (i.e., just the image).

Theorem

Every subset of a countable set is at most countable.

Proof.

Let A be a countable. Let $E \subseteq A$. If E is finite, then E is at most countable and we're done. Suppose E is infinite. Since A is countable, $A \longleftrightarrow \mathbb{N}$. So A may be written in a sequence $\{a_n\}$. Let n_1 be the smallest $n \in \mathbb{N}$ such that $a_{n_1} \in E$. In other words,

$$\{n \in \mathbb{N} | a_n \in E\},$$

because any nonempty subset of $\mathbb{N}$ has a least element. For $k > 1$, let n_k be smallest integer greater than n_{k-1} such that $a_{n_k} \in E$:

$$\{n \in \mathbb{N} | n > n_{k-1}, \quad a_n \in E\}$$

Now we have inductively labeled all the elements of E. This gives a function $\mathbb{N} \to E$. It is bijective, thus showing any subset of a countable set is at most countable. □

Proposition

Consider the following:

- $A \cup (B \cap C) = (A \cap B) \cup (A \cap C)$.
- *(De Morgan's Law) Let* $A_1, A_2, \cdots \subseteq X$. *Then both the following hold:*

$$X \backslash (\bigcup_{i=1}^{\infty} A_i) = \bigcap_{i=1}^{\infty} (X \backslash A_i),$$

$$X \backslash (\bigcap_{i=1}^{\infty} A_i) = \bigcup_{i=1}^{\infty} (X \backslash A_i).$$

note 2: *For* $B \subseteq A$, $A \backslash B = A - B = B^c$.

Theorem

A countable union of countable sets is countable.

Proof.

Let $\{A_n\}$ be a sequence of countable sets (i.e., A_j is countable for each j). Let $A_1 = \{a_{11}, a_{12}, a_{13}, \ldots\}$, $A_2 = \{a_{21}, a_{22}, a_{23}, \ldots\}$, and in general, $A_n = \{a_{n1}, a_{n2}, a_{n3}, \ldots\}$, then we can write

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots \\ a_{21} & a_{22} & a_{21} & \ldots \\ a_{31} & a_{32} & a_{33} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Note that along the matrix diagonals, the sum of the i, j is constant. Consider a_{ij} as an array, we count along the diagonals: $\{a_{11}, a_{21}, a_{12}, a_{31}, a_{22}, a_{13}, \ldots\}$. This is a count of $\bigcup_{i=1}^{\infty} A_i$. □

Challenge: Find an explicit function $f : \mathbb{N} \to \bigcup_{i=1}^{\infty} A_i$.

Theorem

Let A be a countable set. Then $A^n = A \times A \times \cdots \times A$ (n times) is countable.

Recall

$A^n = \{(a_1, \ldots, a_n) | a_i \in A\}$.

Proof.

(Induction) $A^1 = A$, so A^1 is countable. Suppose A^k is countable. Elements of A^{k+1} are of the form (x, y), where $x \in A^k, y \in A$. For each fixed $x_0 \in A^k$, we know

$$\{(x_0, y) | y \in A\} \longleftrightarrow A$$

and thus it is countable. Thus

$$\{(x, y) | x \in A^k, y \in A\}$$

is a countable union of countable sets:

$$\{(x, y) | x \in A^k, y \in A\} = \bigcup_{x_0 \in A^k} \{(x_0, y) | y \in A\}.$$

Therefore A^{k+1} is countable. Hence, the result follows by induction. □

Corollary

$\mathbb{Q}$ is countable.

Proof.

$\mathbb{Z}^2$ is countable by previous theorem. So $\{a/b\}$ is countable. □

Definition

An algebraic number is an element $x_0 \in \mathbb{R}$ such that

$$a_n x_0^n + a_{n-1} x_0^{n-1} + \cdots + a_1 x_0 + a_0 = 0$$

for some $a_0, \ldots, a_n \in \mathbb{Z}$.

Theorem

The set of all algebraic numbers is countable.

Proof.

HW or read in book. □

Theorem

The set $\{0,1\}^\infty$ *is uncountable.*

Note 3: $A^n = \{(a_1, \dots, a_n) | a_i \in A\}$. Thus $A^\infty = \{(a_1, a_2, \dots, a_j, \dots) | a_i \in A\}$.

Proof.

(Cantor) Suppose $\{0,1\}^\infty$ is countable. Consider the following naming:

$$\begin{aligned} x_1 &= x_{11}x_{12}x_{13}\dots & x_{1j} &\in \{0,1\} \\ x_2 &= x_{21}x_{22}x_{23}\dots & x_{2j} &\in \{0,1\} \\ x_3 &= x_{31}x_{32}x_{33}\dots & x_{3j} &\in \{0,1\} \\ &\vdots \end{aligned}$$

In fact, for any $i, j \in \mathbb{Z}^+$, we have $x_{ij} \in \{0,1\}$. Our goal is to construct an $x \in \{0,1\}$ such that $x \neq x_i \, \forall i$.
Let $a_1 \neq x_{11}$, $a_1 \in \{0,1\}$. Let $a_2 \neq x_{22}$... Let $a_i \neq x_{ii}$. Then let $x = (a_1, a_2, a_3, \dots) \in \{0,1\}^\infty$. Observe that for any i, $x \neq x_i$, since $a_i \neq x_{ii}$, by construction. By blasphemy, we get that the set is countable. □

Remark

Same argument works for $\mathbb{R}$. Consider why does this argument not show $\mathbb{Q}$ is uncountable?
Suppose I started counting the reals:

$$x_1 = 0.137294\ldots$$
$$x_2 = 3.123490\ldots$$
$$x_3 = 0.999713\ldots$$

Going down the diagonal works for the reals but not for the rationals because rationals either terminate or repeat. In other words, you cannot prove the result from changing all the diagonals is rational itself.

Other Big Sets

Definition

Let A be a set. The power set of A is the set of all subsets. We write either $P(A)$ or 2^A to denote this set.

Example

Let $A = \{*, \star\}$. $P(A) = \{\{\}, \{*\}, \{*, \star\}, \{\star\}\}$.

Remark

It is important to remember the whole set is in $P(A)$ and the empty set is as well.

Theorem

If A is finite, then $|2^A| = 2^{|A|}$.

Theorem

For any set A,

$$A \not\longleftrightarrow P(A)$$

In other words, a set is not bijectively equivalent to its power set.

Proof.

Suppose $f : A \mapsto P(A)$ is bijective. We will construct a subset $B \subseteq A$, i.e. $B \in P(A)$, such that $B \neq f(a)$ for any $a \in A$. Let $B = \{a \in A | a \notin f(a)\}$. Observe that the element $f(a)$ is a subset of $P(A)$. Suppose $\exists x \in A$, such that $f(x) = B$. If $x \notin B$. Then $x \notin f(x)$ so $x \in B$. If instead $x \in B = f(x)$, so $x \in f(x)$. Thus it follows that $x \notin B$. $\Rightarrow\Leftarrow$. Therefore $B \neq f(a)$ for any $a \in A$. Thus f is not bijective. □

Note 4: $P(A) = \{\text{all functions } f : A \mapsto \{0,1\}\}$. Consider $E \subseteq A$. Let

$$f(a) = \begin{cases} 1 & a \in E \\ 0 & \text{else} \end{cases}$$

This are called membership or characteristic function.

Cardinal Numbers

$0, 1, 2, \ldots, \mathcal{N}_0 (= |\mathbb{N}|), \mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_\infty$. Note that $\mathcal{N}_1$ is the first uncountable size. $\mathcal{N}$'s are indexed by ordinal numbers.

Theorem

(Continuum Hpyothesis) $\mathcal{N}_1 = \mathcal{C}$ *(cardinality of* $\mathbb{R}$*). There are not cardinal numbers between* $\mathcal{N}_0$ *and* $\mathcal{C}$*. Note that this has been proven to be independent of axioms of set theory.*

Metric Space

Question

What is the most general notion of distance?

Definition

A set X is a metric space if there exists a map $d : X \times X \mapsto \mathbb{R}$ such that for all points $p, q \in X$:

- $d(p, q) \geq 0$; *equality* $\iff p = q$.
- $d(p, q) = d(q, p)$.
- $d(p, q) \leq d(p, r) + d(r, q) \; \forall r \in X$.

Here d is called a metric on X. A metric space is a space that comes with a metric defined on it.

Recall

A metric on X must satisfy the following

- $d(a, b) \geq 0$, $(d(a, b) = 0 \iff a = b)$
- $d(a, b) = d(b, a)$
- $d(a, b) \leq d(a, c) + d(c, b)$

Example

Consider the following spaces and corresponding metrics:

- $\mathbb{R}, \quad d(x,y) = |x-y|$
- $\mathbb{R}^n, \quad d(\vec{x},\vec{y}) = \|\vec{x}-\vec{y}\|.$
- $\mathbb{R}^2, \quad d_s(\vec{x},\vec{y}) = \sum_{i=1}^{2} |x_i - y_i|$
- $X =$ *tree (graph),* $\quad d_T(x,y)$ *is the length of a path from two points. Note there is no cycle because then the distance is not well-defined.*
- $X = \{$*genomic sequences*$\}, \quad d_G(x,y) =$ *the number of places they differ.*

Non-Example

On $\mathbb{R}^n$, consider the following metrics that don't work:

- $d(\vec{x}, \vec{y}) = |x_1 - y_1|$
- $d(p, q) = \sum p_i \log \frac{p_i}{q_i}$
- $d(\vec{x}, \vec{y}) = 0$

Definition

Let $x \in X$, $r \in \mathbb{R} \geq 0$. We define $B_r(x) = B(x, r) = \{y \in X | d(x, y) < r\}$. This is called an open ball centered at x of radius R. Also called a neighborhood. The closed ball is given by $\overline{B}(x, r) = \{y \in X | d(x, y) \leq r\}$.

Note 5: Drawings of open and closed balls in $\mathbb{R}^2$ under different metrics not included here.

Idea: Neighborhoods of x are points "close" to x.

Definition

A point $p \in X$ is a limit point of $E \subseteq X$ if every neighborhood of p contains a point q, $q \neq p, q \in E$. Note that a limit point does not need to be a member of the set E.

Example

Let $X = \mathbb{R}$. Let $E = \{\frac{1}{n} | n \in N\}$. We conjecture that 0 is a limit point. This holds as a result of the Archimedean property.

Note 6: If the metric is unspecified, we assume the standard metric, typically Euclidean.

Example

See notes for drawing of $B \subseteq R^2$.

Note 7: p is *not* a limit point of E if there exists a neighborhood U of p such that U contains no point of E other than possibly p.

Logic:

- $\forall$ - universal quantifier
- $\exists$ - existential quantifier

Negation: Suppose $\forall x, p(x)$, where $p(x)$ is some statement. We negate the universal quantifier as such:

$$\neg(\forall x,\ p(x)) \iff \exists x, \neg p(x).$$

Similarly, suppose $\exists x, p(x)$. We negate the existential quantifier as such:

$$\neg(\exists x,\ p(x)) \iff \forall x, \neg p(x).$$

Definition

We say p is an isolated point of E if $p \in E$ and p is not a limit point of E.

Definition

We say p is an interior point if there exists a neighborhood U of p such that $U \subseteq E$.

Note: p is not an interior point if for all neighborhoods U of p, $U \not\subseteq E$. In other words, $\exists x \in U$ such that $x \notin E$.

Definition

A set E in X is open if every point of E is an interior point of E.

Open & Closed Sets

Recall

Consider $E \subseteq X$ where X is metric space is open if every point of E is an interior point.

Definition

A set $E \subseteq X$ is closed if it contains all of its limit points.

Example

Let $X = \mathbb{R}$ and $E = \{x \in \mathbb{R} | a < x \leq b\}$, i.e. the segment $(a, b]$. The interior points of E are (a, b). To see this, let $r = \min\{(d(c, a), d(c, b))\}$. Then $B(c, r) \subseteq E$. The limit points of E are $[a, b]$. Note E is not open because b is not an interior point. Similarly, it is not closed because a is a limit point but not in E.

Theorem

A set $E \subseteq A$, is closed if and only if $A \backslash E$ is open.

Proof.

($\Rightarrow$) Suppose E is closed. So E contains all of its limit points. Our goal is to show that $E^c = A \backslash E$ is open, i.e. all points of E^c are interior points. Let $p \in A \backslash E$. Therefore p is not a limit point of E. Thus there exists a neighborhood U of p such that $U \subseteq A \backslash E$. Thus p is an interior point of $A \backslash E$. Therefore $A \backslash E$ is open.
($\Leftarrow$) Suppose $A \backslash B$ is open. Let $p \in A \backslash E$. We must show p is not a limit point of E. But p is an interior point of $A \backslash E$. So there exists an open neighborhood U of p such that $U \subseteq A \backslash E$. Thus p is not a limit point of E. □

Corollary

A set $E \subseteq X$ is open if and only if E^c is closed.

Theorem

If p is a limit point of $E \subseteq X$, then every neighborhood of p contains infinitely many points of E.

Corollary

A finite set has no limit points.

Proof.

(Constructive) Let $r_1 > 0$. Then there exists $q_1 \neq p$, $q_1 \in E$, such that $q_1 \in B(p, r_1)$. Let $r_2 = d(p, q_1)/2$. Then there exists $q_2 \neq p$, $q_2 \in E$, $q_2 \in B(p, r_2)$. Given $r_{i-1} > 0$ there exists $q_i \neq p$, such that $q_i \in E, q_i \in B(p, r_i)$ where $r_i = \frac{d(p,q_{i-1})}{2}$. Note that $\{q_i\} \subseteq B(p, r_i)$. □

Note: The axiom of choice, which is independent of the other axioms of set theory says if you have an infinite collection of sets, it's possible to assume only element of each if they're nonempty.

Proof.

(Alternate to Theorem 2) Let $r > 0$, $B(p, r)$. Suppose $q_1, \ldots, q_n \in B(p, r)$ where $q_i \neq p$. $B(p, r) \cap E = \{q_1, \ldots, q_n\}$. Let $r' < \min\{\{d(p, q_i) | i = 1, \ldots, n\}\}$. Then $q_i \notin B(p, r')$. But p is a limit point, so there exists $q \in B(p, r')$ such that $q \in E$, $q \neq p$. But $q \in B(p, r)$. Notice that $q \neq q_i$, since $q_i \notin B(p, r')$. Thus $B(p, r) \cap E \neq \{q_1, \ldots, q_n\}$. □

Theorem

Let $\{U_\alpha\}$ be a collection of open sets in X. Then it follows that

$$\bigcup_\alpha U_\alpha$$

is open in X.

Proof.

Let $U = \bigcup_{\alpha} U_\alpha$. Let $p \in U$. Then there exists an α such that $p \in U_\alpha$. But U_α is open. Thus p is an interior point of U_α. Thus there exists $B(p, r) \subseteq U_\alpha$. Hence $B(p, r) \subseteq U$. Thus p is an interior point of U. □

Theorem

Let $U_1, \ldots, U_n$ be open in X. Then

$$\bigcap_{i=1}^{n} U_i$$

is open.

Remark

For intuition, let $U_n = (-\frac{1}{n}, \frac{1}{n})$*. Then it follows that* $\bigcup_{i=1}^{\infty} U_i = (-1, 1)$ *and* $\bigcap_{i=1}^{\infty} U_i = \{0\}$*.*

Proof.

Let $p \in \bigcap_{i=1}^{n} U_i$. Then $p \in U_i$ for all i. But U_i is open for all i. Thus p is an interior point of U_i. Hence, there exists $r_i > 0$ such that $B(p, r_i) \subseteq U_i$. Let $r = \min\{\{r_i | i = 1, \ldots, n\}\}$. Then $B(p, r) \subseteq \bigcap_{i=1}^{n} U_i$. Hence $\bigcap_{i=1}^{n} U_i$ is open. □

Corollary

Let $\{F_\alpha\}$ be closed sets in X. Then

$$\bigcap_\alpha F_\alpha$$

is closed. If $F_1, \ldots, F_n$ are closed, then $\bigcup_{i=1}^n F_i$ is closed.

proof:

$\{F_\alpha\}$ closed. Then $X \backslash F_\alpha$ is open for all X.

$$\bigcap_\alpha = ?$$

Inspect the complement

$$X \backslash (\bigcap_{\alpha} F_{\alpha}) = \bigcup_{\alpha} (X \backslash F_{\alpha})$$

is open. To determine if $\bigcup_{i=1}^{n} F_i$ is closed, we inspect its compliment.

$$X \backslash (\bigcap_{i=n}^{n} F_i) = \bigcap_{i=1}^{n} (X \backslash F_i)$$

is open.

Theorem

$\overline{E}$ is closed. Let $x \in X \setminus \overline{E}$. We want to show x must not be a limit point of $\overline{E}$. We inspect.

To be continued...

Recall

$\bar{E} = E \cup E'$.

Theorem

$E \subseteq X$. $\bar{E}$ is closed.

Proof.

We will show $X\backslash\bar{E}$ is open. Let $x \in X\backslash\bar{E}$. We must show x is in an interior point of $X\backslash\bar{E}$. Since $x \in X\backslash\bar{E}$, we know x is not a limit point of E. Thus there exists an open neighborhood U of x such that U is completely contained in $X\backslash E$. Also note that U contains no limit points of E. Indeed, let $p \in E'$. Suppose $p \in U$. Thus there exists a point $q \neq p$, $q \in U$, $q \in E$. However, we know $U \subseteq X\backslash E$. Thus there does not exist a point $p \in E' \cap U$. Thus $U \subseteq X\backslash E \cap X\backslash E' = X\backslash(E \cap E') = X\backslash\bar{E}$. Thus x is an interior point of $X\backslash\bar{E}$ This holds for all $x \in X\backslash\bar{E}$. Thus $X\backslash\bar{E}$ is open. Therefore $\bar{E}$ is closed. □

Theorem

Let $E \subseteq F \subseteq X$. If F is closed, then $\bar{E} \subseteq F$.

Proof.

Let $p \in E'$. Let U be an open neighborhood of p. There exists $q \neq p$ such that $q \in U$ and $q \in E$. Since $E \subseteq F$, we know $q \in F$. Therefore p is a limit point of F. Since F is closed, $p \in F$. Thus $E' \subseteq F$. Therefore $\bar{E} = E \cup E' \subseteq F$. □

Remark

$$\bar{E} = \bigcap_{\substack{F=\bar{F},\\ E\subseteq F\subseteq X}} F.$$

Question

Is (a,b) open? Consider this interval in $\mathbb{R}$. Consider any $c \in (a,b)$. Indeed we can find a neighborhood of c completely contained in (a,b). Now consider the interval in $\mathbb{R}^2$. In this case it does not. Thus the question is ambiguous.

Definition

Let $Y \subseteq X$ be metric spaces. A subset $E \subseteq Y$ is open relative to Y if and only if there is an open set $U \subseteq X$ such that:

$$E = U \cap Y.$$

Example

Let $Y = (a,b]$ and $X = \mathbb{R}$. Then $(c,b]$ is open in Y for any $c \in (a,b)$.

Compact Sets

Definition

An open cover of $E \subseteq X$ *is a collection* $\{U_\alpha\}$ *of open sets of* X *(*U_α *is open in* X*) such that* $E \subseteq \bigcup_\alpha U_\alpha$.

Example

Consider the following examples:

- *Let* $E = X = \mathbb{R}$. *Consider* $U_n = (n-1, n+1)$.
- *Let* $E = \mathbb{Z}, X = \mathbb{R}$. *Consider* $U_n = (n - \frac{1}{2}, n + \frac{1}{2})$.
- *Let* $E = [0,1], X = \mathbb{R}$. *Consider* $U_n = (\frac{-2}{n}, \frac{2}{n})$.
- *Let* $E = (0,1), X = \mathbb{R}$. *Consider* $U_n = (0, \frac{1}{n}), V_n = (\frac{1}{n}, 1)$. *Consider then* $\{U_n\} \cup \{V_n\}$.
- *Let* $E = \{1,3,4\}, X = \mathbb{R}$. *Let* $U_n = (n - \frac{1}{2}, n + \frac{1}{2})$.

Definition

A set $K \subseteq X$ is compact if and only if every open cover $\{U_\alpha\}$ contains a finite sub cover. $(K \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_k})$

Definition

A set $E \subseteq X$ is compact if every open cover of E has a finite subcover.

f $\{G_\alpha\}$ si an open cover of E and E is compact, then there exists $G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}$ such that $E \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$. Also note that compactness does not follow from the existence of a single finite cover. (A cover with finite number of open sets.) Indeed, since X is open relative to X, every set E in X is covered by a single open set.

Example

Let $E = X = \mathbb{R}$. *Consider* $U_n = (n-1, n+1)$. *We claim* $\{U_n\}$ *has no finite subcover. To see this, suppose to the contrary that it does. In other words,* $\mathbb{R} \subseteq U_{i_1} \cup U_{i_2} \cup \ldots \cup U_{i_l}$, *without loss of generality* $i_1 < i_2 < \cdots < i_l$. *But* $i_1 - 2 \notin \cup_{j=1}^{l} U_{i_j}$. *Thus we have a contradiction and we conclude that* $\mathbb{R}$ *is not compact.*

Example

Let $E = [a, b]$, $X = \mathbb{R}$. *Since* $\mathbb{R} \subseteq \cup U_n$, $[a, b] \subseteq U_n$. *Consider* $\{n \in \mathbb{Z} | n < a\}$. *Let* c *be the largest element. Similarly, there exist* d *which is the smallest integer larger than* b. *Thus* $[a, b] \subseteq U_c \cup U_{c+1} \cup \ldots \cup U_d$. *Thus* $\{U_n\}$ *has a finite subcover. We will prove* $[a, b]$ *is compact.*

Theorem

Suppose $K \subseteq Y \subseteq X$. Then K is a compact subset of Y if and only if K is a compact subset of X.

Proof.

($\Leftarrow$) Let $K \subseteq X$ be compact. Let $\{V_\alpha\}$ be an open cover of K in Y. So $V_\alpha \subseteq Y$ is open, i.e., open $U_\alpha \subseteq X$ such that $V_\alpha = U_\alpha \cap Y$. Then $K \subseteq \cup_\alpha U_\alpha$. So there exists a finite subcover $U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}$. But $K \subseteq Y$. Thus $K \subseteq (U_{\alpha_1} \cap Y) \cup \ldots \cup (U_{\alpha_n} \cap Y) = V_{\alpha_1} \cup \ldots \cup V_{\alpha_n}$. Thus $\{V_\alpha\}$ has a finite subcover.

($\Rightarrow$) Similar by taking the intersection of the open cover with Y. □

Remark

"K is compact" makes sense. In other words compactness is not relative.

Theorem

Let $K \subseteq X$. If K is compact, then K is closed.

Proof.

We prove $X \backslash K$ is open. Let $x \in X \backslash K$. For all $q \in K$, let $r_q < \dfrac{d(x,q)}{2}$. Then $B(x, r_q) \cap B(q, r_q) = \emptyset$. Note that $K \subseteq U_{q \in K} B(q, r_q)$. Furthermore, K is compact so there exist $q_1, \ldots, q_n$ such that $K \subseteq B(q_1, r_{q_1}) \cup \ldots \cup B(q_n, r_{q_n})$. Let $W \subseteq B(q_1, r_{q_1}) \cup \ldots \cup B(q_n, r_{q_n})$. Note that $V = B(x, r_{q_1}) \cap \ldots \cap B(x, r_{q_n})$ does not intersect W. Note $x \in V$. Also note $K \subseteq W$. Thus $V \subseteq X \backslash K$. So x is an interior point of $X \backslash K$. So $X \backslash K$ is open. Thus K is closed. □

Theorem

Let X be compact, $E \subseteq X$. If E is closed then E is compact.

Proof.

Let $\{U_\alpha\} \subseteq X$ be an open cover of E in X. Note that $X \backslash E$ is open. Then $\cup_\alpha U_\alpha \cup \{X \backslash E\}$ covers X. Thus there exists a finite subcover since X is compact of the form $U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}, (X \backslash E)$. Thus $E \subseteq U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}$. So $\{U_\alpha\}$ has a finite subcover. □

Corollary

If F is closed and K is compact then $F \cap K$ is compact.

Proof.

F is closed and K is closed and therefore $F \cap K$ is closed. Therefore $F \cap K$ is compact by previous theorem. □

Goal: Heine-Borel Theorem

Recall

If $F, k \subseteq X$ and if F is closed, and K is compact, then $F \cap K$ is compact.

Theorem

(Nested closed intervals in $\mathbb{R}$ are nonempty.) Let $\cdots \subseteq I_3 \subseteq I_2, \subseteq I_1$ be a sequence of nested closed intervals in $\mathbb{R}$. Let $I_n = [a_b, b_n]$. If $m > n$, then it follows by construction $a_n \leq a_m < b_m \leq b_n$. Then

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

Proof.

Let $x = \sup \{a_i | i \in \mathbb{N}\}$. Note that any b_n is an upper-bound of $\{a_i\}$. Therefore $x \in \mathbb{R}$ exists. So $x \leq b_n$ for all n. Also note that $a_n \leq x$ for all n. Therefore $x \in I_n$ for all n. □

Remark

Same idea works for k-cells in $\mathbb{R}^k$. Note that a k-cell is of the form:

$$[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_k, b_k].$$

Fun Fact: Alternate proof that $\mathbb{R}$ is uncountable. Suppose $\mathbb{R} = \{x_1, x_2, x_3, \dots\}$. Let $I_n = \mathbb{R} \backslash \{x_1, x_2, \dots, x_n\}$. Then $\cdots \subseteq I_3 \subseteq I_2 \subseteq I_1$. So $\{I_n\}$. If I_n is closed. Therefore:

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

Then there exists $z \in \bigcap I_n$, hence a contradiction.

Theorem

Closed intervals in $\mathbb{R}$ are compact. (So are k-cells in $\mathbb{R}^k$.)

Proof.

Let $[a, b] \subseteq \mathbb{R}$. Suppose $[a, b]$ is not compact. Then there exists an open cover $\{G_\alpha\}$ with no finite subcover. Let $c_1 = \frac{a+b}{2}$. Then $\{G_\alpha\}$ covers $[a, c_1]$ and $[c, b]$. At least one has no finite subcover. Without loss of generality, it is $I_1 = [a, c_1]$. Let $c_2 = \frac{a+c_1}{2}$. Then $[a, c_2]$ of $[c_2, c_1]$ has no finite subcover. Call it I_2.

Repeat to obtain $\cdots \subseteq I_3 \subseteq I_2 \subseteq I_1$. Notice these are nested closed intervals with no finite subcover of $\{G_\alpha\}$. Notice also that $|I_n| = 2|I_{n+1}|$. Then there exists a point $x \in \cap I_n$. But $x \in [a, b]$. Thus there exists an α such that $x \in G_\alpha$.

Notice there exists an $r > 0$ such that $B(x, r) \subseteq G_\alpha$. For n large enough, $I_n \subseteq B(x, r)$. But I_n has no finite subcover, so it can't be contained in any finite subcollection of $\{G_\alpha\}$. Hence a contradiction. $\square$

Definition

The set $K \subseteq X$ *is bounded if there exists* $r > 0$ *and* $q \in X$*, such that for all* $p \in K$*,* $d(p, q) < r$*.*

Theorem

Heine-Borel Theorem – In $\mathbb{R}$ *(or* $\mathbb{R}^k$*),* K *is compact if and only if* K *is closed and bounded.*

Proof.

Let $p \in K$. Then $K \subseteq \bigcap_{n \in \mathbb{N}} B(p, n)$, because this covers all of X.

Assume K is compact. Then there exists a finite subcover. Therefore, $K \subseteq B(p, n_1) \cup \cdots \cup B(p, n_l)$. Thus $K \subseteq B(p, r)$ where $r = \max\{n_1, \ldots, n_l\}$. Thus K is bounded. Furthermore, we've already shown that if K is compact, then K is closed.

Conversely, suppose K is closed and bounded. Then there exists $r > 0$ such that $K \subseteq [-r, r]$. Furthermore, $[-r, r]$ is compact. Since K is closed subset of a compact set, K is compact. □

Corollary

Let $K \subseteq \mathbb{R}$. If K is compact, then $\sup k$ exists and $\sup k \in K$.

Proof.

K is bound, hence $\sup k \in \mathbb{R}$. K is closed, hence $K' \subseteq K$. Furthermore, observe that $\sup K \in K'$ because it is a limit point. □

Example

Let $E \subseteq \mathbb{Q}$, $E = \{p \in \mathbb{Q} | 2 < p^2 < 3\}$. In $\mathbb{R}$, is E closed? Is it bounded?

Example

Let A be any set. Suppose A is infinite. Define

$$d(p,q) = \{0 \ if \ p = q, 1 \ else\}.$$

Is $A \subseteq A$ closed? Is A bounded. Yes on both accounts. Notice that $B(p, \frac{1}{2}) = \{p\}$. Thus:

$$A = \bigcup_{p \in A} B(p, \frac{1}{2}).$$

Notice that this cannot be reduced to a finite subcover. Hence, A is not compact.

Goal: Bolzano-Weierstrass Theorem and Connected Sets.

Theorem

A metric space X is compact if and only if every infinite subset E of X has a limit point in X (note that it is not necessarily in E).

Remark

Think of sequences and limit points.

proof:
($\Rightarrow$) Assume X is compact. If no point of X is a limit point of E, then each $q \in E$ has a neighborhood V_q with no other points of E. Notice then that the collection $\{V_q\}$ where $q \in X$ is an open cover of X since no point of X is a limit point of E. But by construction, each V_q contains at most one point of E and $|E| = \infty$. Thus this set has no finite subcover and hence a contradiction since we assumed X was compact.
($\Leftarrow$) (Specifically in $\mathbb{R}^k$, the general proof is HW # 26) Assume every infinite subset E of X has a limit point in X. We will show X is closed and bounded and thus our result will follow from Heine-Borel. Suppose X is not closed. Then there exists a point $z \notin X$ but z is a limit point of X. Choose $E = \{x_n : |x_n - z| < \frac{1}{n}\}$. Note that $E \subseteq X$ because z is a limit point. Then E is infinite, so it has a limit point, but has a limit point at z and no other. This is because if there were another limit point $y \in X$:

$|x_n - y| \geq |z - y| - |x_n - z| \geq |z - y| - \frac{1}{n} \geq \frac{1}{2}(z - y)$. for large n. But this is a contradiction because we've assumed infinite subsets of X have limit points in X. Suppose X is not bounded. Choose y_n such that $|y_n| > n$. Notice that $\{y_n\}$ is an infinite set. But this has no limit point since the points are getting further and further from

Theorem (Bolyano-Weierstrauss)

Every infinite bounded subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

Proof.

If $E \subseteq \mathbb{R}^k$ is infinite and bounded, then E is a subset of a k-cell in $\mathbb{R}^k$. Thus E is an infinite subset of a compact set, and E is bounded. Thus E has a limit point in $\mathbb{R}^k$ and specifically in the k-cell. □

Example

(The Cantor Set) Let K_0 *be* $[0,1]$*. Let* K_1 *be* K_0 *minus the middle third. Let* K_2 *be* K_1 *minus the middle third of the two interval of* K_1*, etc. The cantor set* C *is* $C = \bigcap_{i=1}^{\infty} K_i$*.*
Properties of the Cantor Set:

- C *is nonempty since it is an intersection of nested closed intervals which we showed to be nonempty.*
- C *is closed because arbitrary intersections of closed set is closed.*
- C *is perfect: closed and every point of* C *is a limit point of* C*.*
- C *is bijective with* $\{0,1\}^{\infty}$ *and therefore* C *is uncountable.*
- C *has no interior.*
- C *is totally disconnected.*

Connected Sets

Definition

Let $A, B \subseteq X$, then A and B are separated if $A \cap \bar{B} = \emptyset$ and $\bar{A} \cap B = \emptyset$.

Definition

$E \subseteq X$ is connected if it is not the union of two nonempty separated sets. If E is not connected, it is called disconnected and A and B are a separation of E.

Example

Let $E = \{(x, y) \in \mathbb{R}^2 | x, y \in \mathbb{Q}\}$*. Consider the line* $x = \sqrt{2}$*. We know all the points are on the left and right of this line. In other words, let* $A = \{(x, y) \in E | x < \sqrt{2}\}$ *and* $B = \{(x, y) \in E | x > \sqrt{2}\}$*.*

Theorem

/ *Example.* $[a, b] \subseteq \mathbb{R}$ *is connected.*

Proof.

If not, there exists a separation A, B of $[a, b]$ with $a \in A$. Let $s = \sup A$. S exists and $s \leq b$. Then $s \in \bar{A}$ because supremums are limit points. Thus $s \notin B$. Then $s \in A$ because $A \cup B = [a, b]$. Thus $s \notin \bar{B}$. Thus there exists $\epsilon > 0$ such that $B(s, \epsilon) \subseteq [a, b] \backslash B = A$. Thus $(s - \epsilon, s + \epsilon) \subseteq A$. □

Remark

Rudin proves that $E \subseteq \mathbb{R}$ is connected if and only if it has the "interval property", i.e.,

$$x, y \in E, \; x < z < y \Rightarrow z \in E.$$

Sequences

Recall

A sequence $\{p_n\}$ in X is a function $f:\mathbb{N}\to X$ such that $f(n)=p_n\in X$.

Definition

The sequence $\{p_n\}$ converges in X if there exists $p\in X$ such that for each $\epsilon>0$, there exists $N\in\mathbb{N}$ such that if $n>N$ then $d(p,p_n)<\epsilon$. We say $\{p_n\}$ converges to p. Or equivalently, p is the limit of $\{p_n\}$.

notate: $p_n\to p \iff \lim_{n\to\infty}\{p_n\}=p$.

Definition

If $\{p_n\}$ does not converge to any point in X then it diverges.

Definition

The range of $\{p_n\}$ *is:*

$$\{x \in X | x = p_n \text{ for some } n\}.$$

Definition

We say $\{p_n\}$ *is bounded in* X *if the range of* $\{p_n\}$ *is bounded in* X.

True/False Questions:

(a) $p_n \to p, p_n \to p' \Rightarrow p = p'$. True.

(b) $\{p_n\}$ is bounded $\Rightarrow p_n$ converges. False.

(c) p_n converges $\Rightarrow \{p_n\}$ is bounded. True.

(d) $p_n \to p \Rightarrow p$ is a limit point of the range of $\{p_n\}$. False.

(e) p is a limit point of $E \subseteq X$, then there exists some sequence $\{p_n\}$ such that $p_n \to p$. True.

(f) $p_n \to p \iff$ Every neighborhood of p contains all but finitely many terms in $\{p_n\}$. True.

Remark

"All but finitely many" is equivalent to "Almost all."

proof:

a) Let $\epsilon > 0$. Then there exists $N_1 \in \mathbb{N}$ such that $n > N_1, \ d(p, p_n) < \frac{\epsilon}{2}$. Similarly there exists $N_2 \in \mathbb{N}$ such that $n > N_2, \ d(p', p_n) < \frac{\epsilon}{2}$. Then for $n > \max(N_1, N_2)$:

$$d(p, p') \leq d(p, p_n) + d(p', p_n) < 2\frac{\epsilon}{2} = \epsilon.$$

Thus $d(p, p') < \epsilon$ for all $\epsilon > 0$. Thus $d(p, p') = 0$. Hence $p = p'$.

b) Consider Ex. (4) where there is oscillation between two points.

c) Suppose $p_n \to p$. Since $1 > 0$ (our choice of ϵ), there exists $N \in \mathbb{N}$ such that for $n > N$ implies $d(p_n, p) < 1$. Let $r = \max(1, d(p, p_1), \ldots, d(p, p_n))$. Then $p_i \in B(p, r)$ for all $i \in \mathbb{N}$.

d) Consider Ex. (3) where the sequence is simply one point.

e) $p \in E'$. For all $n \in \mathbb{N}$ such that $p_n \in E$ such that $d(p_n, p) < \frac{1}{n}$ and $p_n \neq p$. There for a sequence $\{p_n\}$. Let $\epsilon > 0$. Choose $N > \frac{1}{\epsilon}$. Then for each $n > N$, we have $d(p, p_n) < \frac{1}{n} < \frac{1}{N} < \epsilon$. Thus this sequences $\{p_n\}$ converges to p as desired.
f) $(\Rightarrow)$ Suppose $p_n \to p$. Given $B(p, \epsilon)$, there exists $N \in \mathbb{N}$ such that when $n > N$, it follows that $d(p, p_n) < \epsilon$, i.e. $p_n \in B(p, \epsilon)$, leaving only finitely many points, p_1 through p_n possible.
$(\Leftarrow)$ For all $\epsilon > 0$, $B(p, \epsilon)$ contains almost all $\{p_n\}$. For $\epsilon > 0$, let

$$m = \max\left(n \in \mathbb{N} | p_n \notin B(p, \epsilon)\right).$$

Then $n > m$ implies $p_n \in B(p, \epsilon)$, i.e. $d(p_n, p) < \epsilon$.

Theorem (*Limit Laws*)

Let $\{s_n\}, \{T_n\}$ be sequences in $\mathbb{C}$ and $s_n \to s$ and $t_n \to t$. Then the following hold:

- $\lim_{n\to\infty}(s_n + t_n) = s + t.$
- $\lim_{n\to\infty}(cs_n) = cs$ *for all* $c \in \mathbb{C}$.
- $\lim_{n\to\infty}(c + s_n) = c + s$ *for all* $c \in \mathbb{C}$.
- $\lim_{n\to\infty}(s_n t_n) = st$ *for all* $c \in \mathbb{C}$.
- $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$ *for all* $c \in \mathbb{C}$.

Proof:

a- Idea: $\left[|(s_n + t_n) - (s + t)| \leq |s_n - s| + |t_n - t|\right]$. Let $\epsilon > 0$. Then there exists $N_1 \in \mathbb{N}$ such that $n > N_1$ implies $d(s_n, s) < \frac{\epsilon}{2}$ and there exists $N_2 \in \mathbb{N}$ such that $n > N_1$ implies $d(t_n, s) < \frac{\epsilon}{2}$. For $n > \max(N_1, N_2)$,

$$\begin{aligned} d(s_n + t_n, s + t) &= |(s_n + t_n) - (s + t)| \\ &= |s_n - s| + |t_n - t| \\ &< \epsilon. \end{aligned}$$

b- Idea: $|cs_n - cs| \leq c|s_n - s|$.

c- Idea: $|s_n t_n - st| = |(s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s)|$. Let $\epsilon > 0$ Choose $k = \max(s, t, 1, \epsilon)$. Then there exists N_1, N_2 such that $n > N_1$ implies $d(s_n, s) < \frac{\epsilon}{3k}$ and similarly if $n > N_2$ implies $d(t_n, t) < \frac{\epsilon}{3k}$. Let $N = \max N_1, N_2$ For $n > N$ we know:

$$\begin{aligned}
|s_n t_n - st| &= |(s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s)| \\
&= \frac{\epsilon^2}{9k^2} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \frac{\epsilon}{9k} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&< \epsilon.
\end{aligned}$$

Subsequences: Some of the terms of the sequence $\{p_n\}$ in order.

Definition

Let $\{p_n\}$ be a sequence in X. Let $\{n_i\}$ be a sequence of natural numbers such that $n_1 < n_2 < n_3 < \ldots$. Then $\{p_{n_i}\}$ is a subsequence of $\{p_n\}$.

Example

Let $\{p_n\} = \{1, \pi, \frac{1}{2}, \pi, \frac{1}{3}, \pi, \ldots\}$. Notice this does not converge. But a subsequence does converge. For example, only the π terms. As do the other terms.

Example

Notice that $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\}$ converges for 1. Notice that any subsequence converges. By property F from last class, almost all terms of the subsequence are contained in any neighborhood of the limit.

Note: If $p_n \to p$, then $p_{n_k} \to p$ for any subsequence $\{p_{n_k}\}$ of $\{p_n\}$.

Question

Must every sequence contain a convergent subsequence?

No. Consider the sequence $\{1, 2, 3, \ldots\}$. There is no convergent subsequence of this sequence.

Theorem

In a compact metric space X, every sequence contains a subsequence converging to a point in X.

Corollary

Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence. (Bolzano-Weierstrass).

Proof.

Let $R = \text{range}(\{p_n\})$. If R is finite, then some $p \in R$ is hit infinitely many times by the sequence by the pigeon-hole principle. Say $\{p_{n_1}, p_{n_2}, \ldots\}$ are those terms. Then this $\{p_{n_k}\}$ converges to p. If R is infinite, then since X is compact, R has a limit point. Use property E to get a subsequence converging to p. □

Cauchy Sequences

Question

How to tell if $\{p_n\}$ converges if you don't know the limit already?

Definition

The sequence $\{p_n\}$ is a Cauchy sequence if for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $m, n > N$ then it is implied that $d(p_m, p_n) < \epsilon$.

Theorem

If $\{p_n\}$ converges, then it is Cauchy.

Proof.

Say $p_n \to p$. Then $\forall \epsilon > 0$ there exists N such that $n > N$ $d(p_n, p) < \frac{\epsilon}{2}$. So for $m, n > N$ we know

$$\begin{aligned} d(p_n, p_m) &\leq d(p_n, p) + d(p_m, p) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

□

Remark

Not every Cauchy sequence converges. To see this, let $X = \mathbb{Q}$. Let p_n be the smallest such that $\frac{m}{n} > \pi$. Then $\{p_n\}$ is Cauchy, but does not converge in $\mathbb{Q}$ (in $\mathbb{R}$ converge to π).

Definition

A metric space X is complete if every Cauchy sequence converges to a point in X.

Theorem

Compact metric spaces are complete.

proof:
Let $\{x_i\}$ be Cauchy in X. Then $\{x_i\}$ has a convergent subsequence. So there exist $\{x_{n_k}\}$ converging to $x \in X$. Fix $\epsilon > 0$. Cauchy implies there exists $N \in \mathbb{N}$ such that $i, j > N$ then $d(x_i, x_j) < \frac{\epsilon}{2}$. By convergence of $\{x_{n_k}\}$, there exists $N' \in N$ such that whenever $n_k > N'$, then $d(x, x_{n_k}) < \frac{\epsilon}{2}$. Let $N'' = \max(N, N')$. Then $i, n_k > N''$ implies

$$\begin{aligned} d(x, x_i) &\leq d(x, x_{n_k}) + d(x_{n_k}, x_i) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence the sequence itself converges to x and therefore X is complete.

Recall

Last time we discussed Cauchy Sequences and Complete Metric Spaces.

Example

$\mathbb{R}$ *is complete but* $\mathbb{Q}$ *is not.*

Theorem

Every metric space (X, d) *has a completion* $(X^\star, \Delta)$. *In other words,* $(X^\star, \Delta)$ *is a complete metric space containing in* X.

Let $X^\star = \{\text{Cauchy sequences in X}\}/\sim$. We will say:

$$\{p_n\} \sim \{p'_n\} \iff \lim_{n\to\infty} d(p_n, p'_n) = 0.$$

Let $P, Q \in X^\star$. Then $P = [\{p_n\}]$, and $Q = [\{p'_n\}]$. We define:

$$\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n).$$

We claim $(X^\star, \Delta)$ is a complete metric space and (X, d) is isometrically embedded in $(X^\star, \Delta)$. In other words, there is an injection $i : X \hookrightarrow X^\star$ such that $d(p, q) = \Delta(i(p), i(q))$.

Example

If $X = \mathbb{Q}$, then $X^\star = \mathbb{R}$. In particular, $X^\star$ is isometrically isomorphic to $\mathbb{R}$. In other there is a distance preserving bijection.

Remark

This is the other construction of $\mathbb{R}$. But Dedekind cuts are more hardcore.

Definition

A sequence $\{s_n\}$ *in* $\mathbb{R}$ *is monotonically increasing if* $s_n \leq s_{n+1}$ *for all* n*. Similarly* $\{s_n\}$ *is monotonically decreasing if* $s_n \geq s_{n+1}$*.*

Theorem

Let $\{s_n\}$ *is monotonic (i.e., either). Then* $\{s_n\}$ *converges in* $\mathbb{R}$ *if and only if it is bounded.*

Proof.

($\Leftarrow$) Suppose $s_n \leq s_{n+1}$ without loss of generality. Let $s = \sup(\text{range}(\{s_n\}))$. Then $s_n \leq s$ for all n.
Let $\epsilon > 0$. Then $s - \epsilon < s$. Thus there exists $N \in \mathbb{N}$ such that $s - \epsilon < s_N \leq s$ since s is a least-upper-bound by construction.
Furthermore, since $s_{n+1} \geq s_n$ for all n. Thus if $n > N$ implies

$$s - \epsilon < s_N \leq s_n \leq s.$$

So $d(s_n, s) < \epsilon$. Therefore $s_n \to s$.
($\Rightarrow$) Already have shown that convergence sequences are bounded. □

Definition

Let $\{s_n\}$ be a sequence in $\mathbb{R}$. We define the upper limit of $\{s_n\}$ in $\mathbb{R} \cup \{\pm\infty\}$ (i.e. the completion of $\mathbb{R}$), is

$$\lim_{n\to\infty} \sup\{s_k | k > n\} = \limsup s_n.$$

Similarly the lower limit of s_n is:

$$\lim_{n\to\infty} \inf\{s_k | k > n\} = \liminf s_n.$$

Theorem (*Book definition*)

Let $\{s_n\}$ *be real. Let* E *be the set of all sub sequential limits of* $\{s_n\}$*. In other words:*

$$E = \{x \in \mathbb{R} \cup \{\pm\infty\} | s_{n_k} \to x \textit{ for some subsequence } \{s_{n_k}\}\}.$$

Let $s^\star = \sup E$, $s_\star = \inf E$. *Then* $s^\star = \limsup s_n$ *and* $s_\star = \liminf s_n$.

Theorem

$s_n \to s \iff \limsup s_n = \liminf s_n = s.$

Theorem

Let $\{s_n\}$ be real. Then

(a) $\limsup s_n \in E$, *using the definition of E above.*

(b) *If* $x > \limsup s_n$, *then* $\exists N \in \mathbb{N}$ *such that* $n > N$ *implies* $s_n < x$. *Moreover,* $\limsup s_n$ *is the only number with this property. It is analogous for* $\liminf s_n$.

proof.(Idea)

Let $s_{n_k} \to t \in E$. Then $\liminf s_{n_k} = \limsup s_{n_k} = t$. But $\{s_{n_k}\} \subseteq \{s_n\}$. Notice:

$$\liminf s_n \leq \liminf s_{n_k} = t = \limsup s_{n_k} \leq \limsup s_n.$$

Thus

$$\liminf s_n \leq \inf E \leq \sup E \leq \limsup s_n.$$

But $\liminf s_n \in E$, $\liminf s_n \in E$. So $\liminf s_n = \inf E$, $\limsup s_n = \sup E$.

Example

Special Sequences. Let $p > 0$.

(a) $\lim_{n\to\infty} \frac{1}{n^p} = 0$.

(b) $\lim_{n\to\infty} p^{\frac{1}{n}} = 1$.

(c) $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

(d) $\lim_{n\to\infty} \frac{n^a}{(1+p)^n} = 0$ *for all* $a \in \mathbb{R}$.

(e) *If* $|x| < 1$, *then* $\lim_{n\to\infty} x^n = 0$.

1. # Series

.

1. # Series

Question

What does this mean?

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6}.$$

2.

Remark

Sum of natural numbers to negative even powers always has a nice form.

Consider also:

$$1 - 1 + 1 - 1 + 1 - 1 + \dots$$

You can associate these differently and get different limits. This tells us we cannot assume associativity in infinite sums.

notation: Let $\{a_n\}$ be a real sequence. Then:

$$\sum_{n=i}^{j} a_n = a_i + a_{i+1} + \cdots + a_j,$$

when $i < j \in \mathbb{N}$.

Definition

The nth partial sum of $\{a_k\}$ is:

$$s_n = \sum_{k=1}^{n} a_k.$$

Remark

$\{s_n\}$ is a real sequence. Sometimes $\{s_n\}$ is called an infinite series.

Definition

If $s_n \to s$ we write

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = s$$

Question

When does an infinite series converge? When its sequence of partial sums converge.

Example

Does $\sum_{n=1}^{\infty} \frac{1}{n}$ *converge?*
Consider partial sums where $s_n = \sum_{k=1}^{n} \frac{1}{k}$*. We use the Cauchy criterion. For* $m < n$*, then*

$$\begin{aligned} d(s_m, s_n) &= s_n - s_m \\ &= s_{m+1} + s_{m+2} + \cdots + s_n \\ &= \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{n} \geq \frac{n-m}{n}. \end{aligned}$$

The inequality comes from observing that all the terms in the sum are less than or equal to $\frac{1}{n}$*. Thus* $s_{2n} - s_n \geq \frac{2n-n}{2n} = \frac{1}{2}$*. Therefore this sequence is not Cauchy. Hence* $\{s_n\}$ *does not converge.*

Theorem (Cauchy Criterion for series)

$\sum a_n$ converges if and only if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $m, n > N$ implies

$$\left| \sum_{k=n}^{m} a_k \right| < \epsilon.$$

Corollary (Divergence Test)

If $\sum a_n$ converges, then $\lim_{n \to \infty} a_n = 0$.

Remark

$\sum \frac{1}{n}$ is called the harmonic series.

Remark

The corollary's converse is not true (counter-example is harmonic series).

Theorem

If $a_n \geq 0$, then $\sum a_n$ converges if and only if partial sums $\{s_n\}$ form a bounded sequence.

Proof.

$\{s_n\}$ is monotonic. Thus bounded implies and is implied by convergence. □

Theorem (Comparison Test)

1. *If $\sum c_n$ converges and $|a_n| \leq c_n$ for almost all n, then $\sum a_n$ converges.*
2. *If $\sum d_n$ diverges to $+\infty$ and $a_n \geq d_n$ for almost all n, then $\sum a_n$ diverges as well.*

Proof.

1. Let $\epsilon > 0$. Since $\sum c_n$ converges, it satisfies Cauchy Criterion. Thus there exists $N \in \mathbb{N}$ such that $m \geq n \geq N$ implies:

$$\sum_{k=n}^{m} c_k \leq \left| \sum_{k=n}^{m} c_k \right| < \epsilon.$$

Thus

$$\left| \sum_{k=n}^{m} a_k \right| \leq \sum_{k=n}^{m} |a_k| \leq \sum_{k=n}^{m} c_k.$$

This follows from the assumption that $|a_n| \leq c_n$ for almost all n so let N be at least larger than the last n for which $c_n < |a_n|$. The resulting inequality satisfies the Cauchy Criterion and thus $\sum a_n$ converges.

2. Follows from (a) via contrapositive. (Also, partial sums form a bounded sequence.) □

Theorem (Geometric Series)

If $|x| < 1$, *then*

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

If $|x| \geq 1$, *then* $\sum x^n$ *diverges.*

Proof.

If $x \neq 1$, let $s_n = \sum_{k=0}^{n} x_k = 1 + x + \cdots + x^n$. Then $s_n = \frac{1-x^{n+1}}{1-x}$ by multiplying both sides by $1 - x$. Thus it follows:

$$\lim_{n\to\infty} s_n = \frac{1}{1-x} \quad \text{if } |x| < 1$$

If $|x| > 1$, then $\{s_n\}$ does not converge. Similarly if $x = \pm 1$, use the divergence test to verify $\{s_n\}$ does not converge. □

Example

$\sum_{n=0}^{\infty} \frac{1}{n!}$ *converges. Notice*

$$\begin{aligned} s_n &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!} \\ &\leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} \\ &< 3. \end{aligned}$$

Thus it is bounded and since each term is nonnegative, it is monotonically increasing. Thus it converges.

Definition

$e = \sum_{n=0}^{\infty} \frac{1}{n!}$.

Remark

$$\begin{aligned} e - s_n &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \dots \\ &= \frac{1}{(n+1)!}(1 + \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \dots) \\ &= \frac{1}{n!n} \end{aligned}$$

Theorem

$e \notin \mathbb{Q}$.

Proof.

Suppose $e = \frac{m}{n}$ for $m, n > 0$. Then:

$$0 \leq \underbrace{n!(e - s_n)}_{\in \mathbb{Z}} < \frac{1}{n} < 1$$

□

Remark

$e = \lim_{n\to\infty}(1 + \frac{1}{n})^n$.

Recall

A series converges if its partial sums converge.

Example

$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$ *converges because partial sums converge.*

Theorem (Cauchy)

If $a_1 \geq a_2 \geq a_2 \cdots \geq 0$, *i.e. monotonically decreasing, then* $\sum a_n$ *converges if and only if* $\sum 2^k a_{2k} = a_1 + 2a_2 + 4a_4 + 8a_4 + \dots$ *converges.*

Proof.

Compare $s_n = a_1 + \cdots + a_n$ and $t_k = a_1 + 2a_2 + \cdots + 2^k a_{2^k}$.
Consider the following grouping of the finite sum:

$$s_n = (a_1) + (a_2 + a_3) + (a_4 + \cdots + a_7) + \cdots + a_n$$
$$t_k = (a_1) + (a_2 + a_2) + (a_4 + \cdots + a_4) + \cdots + (a_{2^k} + \cdots + a_{2^k})$$

If $n < 2^k$, then $s_n < t_k$. If $n > 2^k$, then $2s_n > t_k$. Compare then:

$$2a_1 + 2a_2 + 2(a_3 + a_4) + \ldots$$
$$a_1 + (a_2 + a_2) + 4a_4 + \ldots$$

So both series diverge or converge together. □

Theorem

Consider $\sum \frac{1}{n^p}$. *Claim is that this converges if* $p > 1$ *and diverges if* $p \leq 1$.

Proof.

If $p \leq 0$, terms do not go to zero, so the series diverges. If $p > 0$, look at:

$$\sum_k 2^k \frac{1}{(2^k)^p} = \sum_k 2^{(1-p)k},$$

which is geometric. Thus it converges if and only if $2^{1-p} < 1$. This only happens when $1 - p < 0$ and hence $p > 1$. $\square$

Remark

We were able to turn a harmonic-like series into a geometric-like series.

Theorem (Root Test)

Given $\sum a_n$ *Let* $\alpha = \lim_{n\to\infty} \sqrt[n]{|a_n|}$. *If* $\alpha < 1$, *then* $\sum a_n$ *converges. If* $\alpha > 1$, *then* $\sum a_n$ *diverges. If* $\alpha = 1$, *then the test is inconclusive. However this is unsatisfactory. So let* $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$, *which always exists (though it may be infinity).*

Proof.

(By comparison with the geometric series) Suppose $\alpha < 1$. Remark that if a sequence passes a limsup, then it passes it for only finitely many terms. Choose $\alpha < \beta < 1$. By the definition of limsup there exists N such that $n > N$ implies $\sqrt[n]{|a_n|} < \beta$. So $|a_n| < \beta^n$. But $\sum b^n$ converges. Therefore, the sum of the a_n converges as well.
If $\alpha > 1$, then there exists a subsequence of $\sqrt[4]{|a_n|}$, say $\sqrt[a_k]{|a_{n_k}} \to \alpha > 1$. So $\sqrt[a_k]{|a_{n_k}} > 1$ for k large enough. This implies $|a_{n_k}| > 1$ and therefore the terms do not go to zero and thus the sequence does not converge.
If $\alpha = 1$, notice $\sum \frac{1}{n}$ diverges but $\sum \frac{1}{n^2}$ converges but in both cases $\alpha = 1$ and therefore this case is inconclusive. □

Theorem (Ration Test)

$\sum a_n$ converges of $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$, diverges if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for n large enough.

Proof.

If the ratio is always bigger than 1 so the series doesn't converge. If the ratio is smaller than 1, then we know $\left|\frac{a_{n+1}}{a}\right| < \beta < 1$ for some large N. So $a_{N+k} < \beta a_{N+k-1} < \beta^2 a_{N+k-2} < \cdots < \beta^k a_n$. So compare:

$$\sum_{k=0}^{\infty} a_{N+k} \quad a_n \sum_{k=0}^{\infty} \beta^k.$$

□

Definition

A power series is of the form $\sum_{k=0}^{\infty} c_n z^n$ *where* $c_n \in \mathbb{C}$.

Theorem

Let $\alpha = \limsup \sqrt[n]{|c_n|}$. *Let* $r = \frac{1}{\alpha}$. *Then the power series converges if* $|z| < R$ *and diverges* $|z| > R$. *We call* R *the radius of convergence.*

Proof.

Use the root test so consider $\limsup_{n\to\infty} \sqrt[n]{|c_n z^n|} = \limsup_{n\to\infty} |z| \sqrt[n]{|c_n|} = |z| \limsup_{n\to\infty} \sqrt[n]{|c_n|}$. Notice that this less than 1, and thus converges, when $|z|$ is less than 1 over the limsup. □

Definition

A series converges absolutely if $\sum |a_n|$ *converges.*

Theorem

$\sum a_n$ *converges absolutely implies* $\sum a_n$ *converges.*

Proof.

$\left|\sum_{k=n}^{m} a_k\right| \leq \sum_{k=n}^{m} |a_k| < \epsilon$, by the Cauchy Criterion since $\sum |a_k|$ converges. □

Example

If a series converges, it does not necessarily converge absolutely. Consider the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$, *which converges by alternating series test but does not converge absolutely.*

Question

If the terms in a convergent series are rearranged, must it converge to same sum? Not all the time, but it does if the series converges absolutely.

Theorem (Riemann)

If a series $\sum a_n$ converges but not absolutely, then we can form a rearrangement that has any limsup and liminf you'd like.

Example

Rearrange $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ *to converge to 4. We want to the partial sums to converge to 4. Consider the positive terms in sequence that sum to at least 4 (notice the positive terms diverge). Then use a s many negative terms you need to go back, as many positive terms you need to go forward, et cetera. These partial sums converge to 4.*

2 **Limits & Continuity**

2. **Limits & Continuity**

.

Recall

We have studied $\lim_{n\to\infty} a_n = l$. *We now want a notion of* $\lim_{x\to p} f(x) = p$.

2. # Limits & Continuity

Recall

We have studied $\lim_{n\to\infty} a_n = l$. *We now want a notion of* $\lim_{x\to p} f(x) = p$.

Definition

Let X, Y be metric spaces, with $E \subseteq X$ and $p \in X$ is a limit point of E. Let f be a map $f : E \to Y$. In other words the domain of f is E. Note p need not be a point of E. We say $f(x) \to q$ as $x \to p$ or equivalently

$$\lim_{x \to p} f(x) = q$$

if there exists $q \in Y$ such that for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in E$, $0 < d_x(x, p) < \delta$ implies $0 < d_y(f(x), q) < \epsilon$. Notice this requires x and p are not the same point, and therefore f need not be defined at p.

Theorem

$\lim_{n \to p} f(x) = q$ *if and only if for all sequences $\{p_n\}$ in E, where for each point $p_n \neq p$, and $p_n \to p$, we have $f(p_n) \to q$.*

Proof.

$(\Rightarrow)$ Let $\epsilon > 0$. We want to find N for any sequence $\{f(p_n)\}$. Because $\lim_{x \to p} f(x) = q$, there exists $\delta > 0$. Also, we know $\{p_n\} \to p$. So there exists N such that for all $n > N$ it follows that $d_x(p_n, p) < \delta$ (by replacing ϵ by δ). Thus $d_y(f(p_n), q) < \epsilon$.

$(\Leftarrow)$ If $\lim_{x \to p} f(x) \neq q$, then there exists $\epsilon > 0$ such that for every $\delta > 0$, there is some point $x \in E$ such that $0 < d_x(x, p) < \delta$ but $0 < d_y(f(x), q) < \epsilon$. We find a bad sequence. Use $\delta_n = \frac{1}{n}$ and choose x_n for each δ_n. Then $x_n \to p$ because the distances from p converges to zero. But $f(x_n) \not\to q$ by our assumption. This shows false implies false, and hence the contrapositive of what we're trying to show. □

Note: The following come from the previous theorem:

- $\lim_{x \to p} f(x)$ is unique
- For all valued functions, sums of limits are limits of sums, and analogously for products and quotients when defined.

Definition

Let X, Y be metric spaces. Let $p \in E \subseteq X$ and $f : E \to Y$. Notice p is necessarily in the domain of f. We say f is continuous at p if for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in E$ such that $d_x(x, p) < \delta$ implies $d_y(f(x), f(y)) < \epsilon$. We say f is continuous on E if f is continuous at all points $p \in E$.

Theorem (Idea: continuous functions preserve limits)

If p is a limit point of E, then f is continuous at p if and only if $\lim_{x\to p} f(x) = f(p)$. Hence f is continuous on E if and only if for all convergence $\{x_n\}$ in E, $\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$.

Theorem

A function $f : X \to Y$ is continuous if and only if for all open sets $U \subseteq Y$, $f^{-1}(U)$ is open in X.

Proof.

($\Rightarrow$) Consider $p \in f^{-1}(U)$. We have to show p is an interior point. Note $f(p) \in U$. U is open by assumption, so $f(p)$ is an interior point of U. So there exists $\epsilon > 0$ such that $B(f(p), \epsilon) \subseteq U$. So there exists $\delta > 0$ such that $f(B(p, \delta)) \subseteq B(f(p), \epsilon)$ because f is continuous. Furthermore, notice $B(p, \delta) \subseteq f^{-1}(U)$ and thus we see p is an interior point.

($\Leftarrow$) Fix $p \in X$. Choose $\epsilon > 0$. Let $U = B(f(p), \epsilon)$. Then $f^{-1}(U)$ is open. Then $p \in f^{-1}(U)$ is an interior point. So there exists $\delta > 0$ such that $B(p, \delta) \subseteq f^{-1}(U)$ i.e. $d_x(x, p) < \delta$ implies $d_y(f(x), f(p)) < \epsilon$. □

Recall

A function $f : X \to Y$ is continuous if and only if for all open subsets $U \subseteq Y$ implies $f^{-1}(U)$ is open in X.

Example

Consider $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$. Notice $f^{-1}((1,2)) = (-\sqrt{2}, -1) \cup (1, \sqrt{2})$, $f^{-1}((-1,-2)) = \emptyset$, and $f^{-1}((-1,1)) = (-1,1)$.

Corollary

A function $f : X \to Y$ is continuous if and only if for all closed subsets $K \subseteq Y$ implies $f^{-1}(K)$ is closed in X.

Proof.

Claim: $X \backslash f^{-1}(K) = f^{-1}(Y \backslash K)$. Notice:

$$
\begin{aligned}
f^{-1}(Y \backslash K) &= \{x \in X | x \in f^{-1}(Y \backslash K)\} \\
&= \{x | f(x) \in Y \backslash K\} \\
&= \{x | f(x) \notin K\} \\
&= \{x | x \notin f^{-1}(K)\} \\
&= X \backslash f^{-1}(K).
\end{aligned}
$$

Assume continuous. Let $K \subseteq Y$ be closed. Then $Y \backslash K$ is open. Hence $f^{-1}(Y \backslash K)$ is open. Thus $X \backslash f^{-1}(K)$ is open. Hence $f^{-1}(K)$ is closed. □

Definition

Let X be a set. Let $\mathcal{T}$ be a collection of subsets of X such that:

1. $\emptyset \in \mathcal{T}$
2. $X \in \mathcal{T}$
3. *If* $U_\alpha \in \mathcal{T}$ *for all* $\alpha \in I$*, then* $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$
4. *If* $U_{\alpha_1}, \ldots, U_{\alpha_n} \in \mathcal{T}$ *implies* $U_{\alpha_1} \cap \cdots \cap U_{\alpha_n} \in \mathcal{T}$

Then $\mathcal{T}$ *is a topology on* X*,* $(X, \mathcal{T})$ *is a topological space,* $U \in \mathcal{T}$ *are called open sets.*

Theorem

(Continuous maps preserve compactness) Let $f : X \to Y$ *be continuous. If* X *is compact, then* $f(X)$ *is compact.*

Proof.

We show any open cover of $f(X)$ has a finite subcover. Let $\{U_\alpha\}$ be an open cover of $f(X)$. Note that $f^{-1}(\bigcup_\alpha U_\alpha) = X$. Notice $f^{-1}(\bigcup_\alpha U_\alpha) = \cup f^{-1}(U_\alpha)$. Also, $f^{-1}(U_\alpha)$ is open for all α since f is continuous. Thus $f\{(U_\alpha)\}$ is an open cover of X. By assumption, X is compact, so there exists a finite subcover of $f^{-1}(U_{\alpha_1}), \ldots, f^{-1}(U_{\alpha_n})$. So $X = f^{-1}(U_{\alpha_1}) \cup \cdots \cup f^{-1}(U_{\alpha_n})$. Hence $f(X) = U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$. This is a finite subcover. □

Corollary

Let $f: X \to \mathbb{R}^k$ be continuous. If X is compact, then $f(X)$ is closed and bounded.

Corollary

Let $f : X \to \mathbb{R}$ be continuous. If X is compact, then $f(x)$ achieves both its maximum and minimum values.

Proof.

Let $M = \sup\{f(x) | x \in X\}$, $m = \inf\{f(x) | x \in X\}$. Then m, M exist because $f(X)$ is bounded. Also $m, M \in f(X)$ because $f(X)$ is closed. □

Remark

We say f is a bounded function if $f(X)$ is bounded.

Question

Suppose $f : X \to Y$ is a bijection (i.e., onto and injective) and therefore $f^{-1} : Y \to X$ exists. If f is continuous, does that imply f^{-1} is continuous? No!

Theorem

Let $f : X \to Y$ be a bijection and continuous. Then if X is compact, then $f^{-1}(Y)$ is also continuous.

Proof.

We must show that for any open $U \subseteq X$ that it follows that $(f^{-1})^{-1}(U)$ is open in Y. But $(f^{-1})^{-1} = f$ since f is bijective. Let U be open in X. Then $X \backslash U$ is closed. Hence $X \backslash U$ is compact. Thus $f(X \backslash U)$ is compact. Also $f(X \backslash U) = Y \backslash f(U)$. So $Y \backslash f(U)$ is compact. Thus $Y \backslash f(U)$ is closed. Thus $f(U)$ is open. □

Note: If $f : X \to Y$ is bijective where f, f^{-1} are continuous, then f is called *homeomorphism*.

Theorem (Continuous maps preserve connectedness as well)

Let $f : X \to Y$ be continuous. If X is connected, then its image is connected.

Proof.

Prove the contrapositive. If $f(X)$ is not connected. Then we know $f(X) = U \cup V$ where U, V are open and $U \cap V = \emptyset$. But then $X = f^{-1}(U) \cup f^{-1}(V)$. We know these are disjoint because if $x \in f^{-1}(U) \cap f^{-1}(V)$, then $f(x) \in U$ and $f(x) \in V$, but we assumed they're disjoint. Furthermore, they're open in X. Therefore X is disconnected. □

Theorem (Intermediate Value Theorem)

Let $f : [a, b] \to \mathbb{R}$ be continuous. If $c \in (f(a), f(b))$, then there exists $x \in (a, b)$ such that $f(x) = c$.

Proof.

Without loss of generality let $f(a) < c < f(b)$. If not, then $(f(a), f(c)), (f(c), f(b))$ form a separation of the image of the map $f((a, b])$. But (a, b) is connected and we know the continuous image of a connected set is also connected. □

Definition

A function $f : X \to Y$ is uniformly continuous if for all $\epsilon > 0$ there exists $\delta > 0$ such that:

$$d_X(p, x) < \delta \implies d_Y(f(p), f(x)) < \epsilon.$$

Remark

The same δ must work for all p. If f is just continuous, then δ may depend on p.

Example

Consider $f(x) = \frac{1}{x}$. *Let* $\epsilon > 0$. *Choose any* $\delta > 0$. *Then there exists* $n \in \mathbb{N}$ *such that* $\frac{1}{n} < \epsilon$. *Choose* $p \in (-\frac{1}{2n}, 0), q \in (0, \frac{1}{2n})$. *Thus:*

$$d(p,q) < \frac{1}{2n} - \left(-\frac{1}{2n}\right) = \frac{1}{n} < \delta.$$

But $d(f(p), f(q)) = \frac{1}{q} - \frac{1}{p} > 2n + 2n = 4n$. *So there exists* n' *such that* $4n' > \epsilon$. *Choose* $n'' = \max(n, n')$. *Then there exists* p, q *such that* $d(p,q) < \delta$ *but* $d(f(q), f(p)) > \epsilon$. *Thus* $f(x)$ *is not uniformly continuous.*

Theorem

If f is uniformly continuous, then f is continuous.

Proof.

Exercise for reader. □

Theorem

Let $f : X \to Y$ be continuous. If X is compact, then f is uniformly continuous.

Proof.

Let $\epsilon > 0$. Our goal is to find a δ that works for all p. Each $x \in X$ has some δ_x-ball such that

$$d(x, p) < \delta_x \implies d(f(p), f(x)) < \epsilon.$$

We want to find a δ such that $d(p, q) < \delta$, or in other words, p and q are in the same δ_x ball. Then $d(f(p), f(q)) < d(f(p), f(x)) + d(f(x), f(q)) < \epsilon$. Use $\frac{\epsilon}{2}$ as δ. □

Lemma (Lebesgue's Covering Lemma)

If $\{U_\alpha\}$ is an open cover of a compact metric space X, then there exists $\delta > 0$ such that for all $x \in X$, $B(x, \delta) \subseteq U_\alpha$ for some α. (δ is called the Lebesgue number of the open cover $\{U_\alpha\}$.)

Proof.

Since X is compact, there exists a finite cover $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$. For any $K \subseteq X$, define:

$$d(x, K) = \inf\{d(x, y) | y \in K\}.$$

We claim $d(x, k)$ is a continuous function of x. Let $f(x) = \frac{1}{n} \sum_1^n d(x, X \backslash U_{\alpha_i})$. Then f is continuous on X, it achieves its minimum value, δ. For all x, $f(x) \geq \delta \implies$ at least one $d(x, X \backslash U_{\alpha_i}) \geq \delta$. Thus $B(x, \delta) \subseteq U_{\alpha_i}$. □

Discontinuities

Let $f: (a,b) \to \mathbb{R}$. For all $\{t_n\}$ in (a,x) with $t_n \to x$, then $f(t_n) \to q$. We write $f(x^-) = q$ or $\lim_{t \to x^-} f(t) = q$. Similarly, if for all $\{t_n\}$ in (x,b) with $t_n \to x$, we have $f(t_n) \to q'$ and write $f(x^+) = q'$ or $\lim_{t \to x^+} f(t) = q'$.

Theorem

$\lim_{t \to x} f(x) exists \iff f(x^+) = f(x^-)$.

Remark

If f is discontinuous at x, but $f(x^+)$, $f(x^-)$ both exist, then we say f has "discontinuity of first kind" or "simple". Otherwise, it is the second kind. An example of the second kind is:

$$f(x) = \begin{cases} 0 & x \leq 0 \\ \sin(\frac{1}{x}) & x > 0 \end{cases}$$

Note that $f(x)$ is called the topologist's sine curve.

Example

Consider

$$f(x) = \begin{cases} 0 & x = 0 \\ x\sin(\frac{1}{x}) & x > 0 \end{cases}$$

This has no discontinuity at zero.

Example

(Dirichlet function)

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

This function has the second kind of discontinuity at all points.

Example

$$f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Example

$$f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \\ 0 & else \end{cases}$$

This function is continuous at all irrationals and has simple discontinuities at all rationals. Thus there are an uncountable number of discontinuities.

Theorem

If $f : \mathbb{R} \to \mathbb{R}$ is monotonic, then it has no discontinuities of the second kind.

Sequences & Series of Functions (Chapter 7)

Definition

For each $n \in \mathbb{N}$, let $f_n : X \to Y$ be a function. Then $\{f_n\}$ is sequence of functions. If $\{f_n(a)\}$ converges for each element $a \in X$ we say the sequence converges point wise to $f(x) = \lim_{n\to\infty} f_n(x)$.

Definition

Let $f_n : X \to Y$ be a sequence of functions. If $\sum f_n(x)$ converges for each $x \in X$, we say the series converges point wise to $f(x) = \sum_{n=1}^{\infty} f_n(x)$.

Question

If $f_n \to f$ point wise and each f_n is continuous for all $n \in \mathbb{N}$, is f continuous?

Example

Let $f_n(x) = \frac{x}{n}$. Notice this converges to $f(x) = 0$.

Example

Let $f_n(x) = x^n$ with $f : [0, 1] \to \mathbb{R}$. This converges to

$$f(x) = \begin{cases} 1 & x = 1 \\ 0 & x \neq 1 \end{cases}$$

Notice this example shows that point wise convergence does not imply continuity.

Note: Area may not be preserved by point wise convergence.

Example

Let $f_n(x) = \frac{1}{n}\sin(n^2 x)$*. Notice that this converges point wise to zero.*

Note: Derivative may not be preserved as well.

Example

Let $f_m(x) = \lim_{n\to\infty}(\cos(m!\pi x))^{2n}$*. If* $m!x \in \mathbb{Z}$*, then* $f_m(X) = 1$*. If* $x \in \mathbb{Q}$*, then* $x = \frac{p}{q} \implies m!x \in \mathbb{Z}$ *for* $m \geq q$*. Thus for* $x \in \mathbb{Q}$*, then* $\lim_{m\to\infty} f_m(x) = 1$*. For* $x \notin \mathbb{Q}$*,* $f_m(x) = 0$*. So:*

$$f_m(x) \to f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Remark

We're seting $f(x) = \lim_{n\to\infty} f_n(x)$. *Recall that* f *is continuous at* x *if and only if* $\lim_{t\to x} f(t) = f(x)$. *Thus if* $f(x)$ *is continuous:*

$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t).$$

Note that normally cannot reorder limits.

Example

$S_{m,n} = \frac{m}{m+n}$ *Fix* n, *then* $\lim_{m\to\infty} S_{m,n} = 1$ *and therefore* $\lim_{n\to\infty}\lim_{m\to\infty} S_{m,n} = 1$. *But if we fix* m, $\lim_{n\to\infty} S_{m,n} = 0$ *and therefore* $\lim_{m\to\infty}\lim_{n\to\infty} S_{m,n} = 0$.

Definition

We say $f_n \to f$ *uniformly on* $E \subseteq X$ *if for all* $\epsilon > 0$*, there exists* $N \in \mathbb{N}$ *such that* $n > N \implies \|f_n - f\| < \epsilon$*, where* $\|f\| = \sup_{x \in E} |f(x)|$.

Remark

Note that $\|f_n - f\| < \epsilon$ *if and only if* $|f_n(x) - f(x)| < \epsilon$ *for all* $x \in E$.

Remark

The set $\{f : E \to Y\}$ *is a metric space with metric* $\|\cdot\|$*.*
$\mathcal{C}_b(E) = \{f : E \to Y\}$ *where* f *is continuous and bounded.*

Note $\mathcal{C}_b(E)$ is complete.

Theorem

(Cauchy Criterion) $f_n \to f$ *uniformly* $\iff \forall \epsilon > 0 \exists N \in \mathbb{N}$ *such that* $m, n > N \implies |f_n(x) - f_m(x)| < \epsilon$.

Theorem (Weierstrauss 1872)

$$f = \sum b^n \cos(a^n \pi x)$$

with $0 < b < 1$, $a \in 2\mathbb{Z} + 1$, $ab > 1 + \frac{3\pi}{2}$. *Then* f *is continuous everywhere but differentiable nowhere.*

Theorem

Let $\{f_n\}$ *be continuous. If* $f_n \to f$ *uniformly, then* f *is continuous.*

Proof.

$$|f(x) - f(y)| \leq |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f_k(y) - f(y)|$$

Fix x. We will show continuity at x. Let $\epsilon > 0$. Choose f_k such that $\|f_k - f\| < \frac{\epsilon}{3}$ (by uniform convergence). Thus the first and third terms are less than $\frac{\epsilon}{3}$. Also f_k being continuous implies there exists $\delta > 0$ such that $|x - y| < \delta$, that implies the second term is less than $\frac{\epsilon}{3}$. Therefore, we've found a $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < 3\frac{\epsilon}{3} = \epsilon$. □