N° D'ORDRE 40

UNIVERSITE CADI AYYAD
FACULTE DES SCIENCES
SEMLALIA-MARRAKECH

****

THESE

Présentée à la Faculté, pour obtenir
Le grade de Docteur
(Option : Mathématiques Appliquées)

CONTRIBUTION A L' ETUDE DES ASPECTS QUANTITATIF ET QUALITATIF POUR UNE CLASSE D' EQUATIONS DIFFERENTIELLES A RETARD INFINI, EN DIMENSION INFINIE

Par

HASSANE BOUZAHIR
(Url : www.geocities.com/hbouzahir)

Soutenue le 05 avril 2001 devant la commission d'examen

Président **A. AITOUASSARAH :** Professeur, Université Cadi Ayyad, Marrakech

Examinateurs **M. ADIMY :** Professeur, Université de Pau et des Pays de l'Adour, Pau (France)
E. AIT DADS : Professeur, Université Cadi Ayyad, Marrakech
Rapporteur
H. BOUSLOUS : Professeur, Université Cadi Ayyad, Marrakech
K. EZZINBI : Professeur Habilité, Université Cadi Ayyad, Marrakech
Encadrant
My L. HBID : Professeur, Université Cadi Ayyad, Marrakech
A. RHANDI : Professeur, Université Cadi Ayyad, Marrakech

A la mémoire de mon père
(rahimah ALLAH)

A ma très chère maman

A mon frère aîné Abdelhay

A toutes mes soeurs

A mes petits frères

A tous mes proches

A tous ceux qui me sont chèrs.

" Ce ne sont pas des fleurs printanières sujettes au changement des saisons, mais étant cueillies dans les plus beaux parterres de la mathématique, ce sont plutôt des amarantes qui ne flétriront jamais "

(Blaise Pascal)

REMERCIEMENTS

Au terme de cette thèse, il m'est très agréable de m'acquitter d'une dette de reconnaissance auprès de toutes les personnes dont l'intervention a favorisé son aboutissement.

Je voudrais d'abord exprimer ma grande gratitude à mon superviseur le Professeur **Khalil Ezzinbi** pour son excellent encadrement, son constant encouragement et le fait d'être la meilleure source continue de mes idées. C'est avec amitié et confiance qu'il m'a initié à la carrière de recherche. Je le remercie de m'avoir bien accueilli dans son bureau et de m'avoir transmis toutes ses expériences.

Je tiens à exprimer ma grande gratitude au Professeur **Abderrahman Aitouassarah** qui m'a fait l'honneur et le plaisir de présider le jury.

J'adresse mes sincères remerciements aux Professeurs **Elhadi Ait Dads** de l'Université Cadi Ayyad**, Ovide ARINO** de l'Université de Pau et des Pays de l 'Adour, **Jack Kenneth HALE** de Georgia Institute of Technology **et Jianhong WU** de York University de me faire le grand honneur d'avoir accepter de rapporter mon travail. Je leurs exprime aussi ma grande gratitude pour leurs précieux conseils et discussions sur mon sujet de recherche.

Je suis très sensible à l'amitié et la gentillesse que j'ai trouvé auprès des Professeurs **Mostafa Adimy** de l'Université de Pau et Pays de l'Adour et **Rachid Benkhalti** de Pacific Lutheran University. Leurs aides m'ont fait profiter des discussions et collaborations fructueuses qu'on a eu ensemble.

L'amitié et la sympathie que le Professeur **Abdelaziz Rhandi** me témoigne me flattent toujours. Je lui exprime ma grande gratitude pour son aide.

Les Professeurs **Hammadi Bouslous** et **Moulay Lhassan Hbid** ont toujours des conseils judicieux à me prodiger. Qu'ils trouvent ici l'expression de ma profonde reconnaissance pour leurs aides et leurs soutiens.

Mes remerciements vont aussi aux Professeurs **Abdelilah Dahlane**, **Mohammed Elalaoui Talibi**, **Mohammed Erraoui, Mohammed Khaladi**, **Salah Labhala, Lahcen Maniar**, **Brahim Sadik, Youssef Ouknine** pour leurs sens humain et leur soutien.

Je saisis l'occasion pour remercier les collègues avec qui j'ai partagé des moments de joie comme ceux de peine tout au long de nos années de troisième cycle. Chacun de nous a été une source d'idées et informations pour les autres. Je remercie également mes autres camarades étudiants et enseignants du département de mathématiques.

Je témoigne aussi la gentillesse des personnels du département de mathématiques, service de troisième cycle, service d'intendance, service des bourses, service de relations extérieures, buvette et veilleurs. Je les remercie beaucoup.

Je n'oublierais jamais de prier la miséricorde pour mon père **Elarbi Bouzahir** qui a contribué puissamment à la formation de mon esprit et n'a épargné aucun effort pour que je mène à bien mes études. Qu'Allah aille son âme.

En fin, j'adresse mes remerciements à mes proches et amis pour leurs aides morales et matérielles durant la préparation de cette thèse. Mes remerciements les plus vifs vont à mon frère **Abdelhay Bouzahir** qui a été le sponsor principal de mes études de troisième cycle.

Avant Propos

-Nom et Prénom de l'auteur : BOUZAHIR Hassane

-Intitulé du travail :

Contribution à l'Etude des Aspects Quantitatif et Qualitatif pour une Classe d'Equations Différentielles à Retard Infini, en Dimension Infinie

-Nom et Prénom du directeur de recherche : EZZINBI Khalil

-Laboratoire où les travaux ont été réalisés : (Intitulé et institution)

Laboratoire des Processus Stochastiques et Systèmes Dynamiques,
Faculté des Sciences Semlalia, Université Cadi Ayyad, Marrakech (Maroc)

-Laboratoire avec lequel il y a eu collaboration pour ce travail :

Laboratoire de Mathématiques Appliquées,
IPRA de l'Université de Pau et des Pays de l'Adour, Pau (France).

-Date de commencement de ce travail : Novembre 1997

-Rapporteurs autres que l'encadrant :

El Hadi AIT DADS, Professeur à la Faculté des Sciences Semlalia, Université Cadi Ayyad, Marrakech (Maroc)

Ovide ARINO, Chercheur à l'Institut de Recherche pour le Développement (IRD), Bondy, Paris (France) et Professor at the Department of Mathematics at Brigham Young University (USA)

Jack Kenneth HALE, Regents Professor Emeritus of Applied Mathematics at the Georgia Institute of Technology, Atlanta (USA)

Jianhong WU, Professor at the Department of Mathematics and Statistics, York University, North York, Ontario (Canada)

-Cadres de coopération-soutien financier :

-Bourse marocaine de troisième cycle
-Une bourse junior d'un mois du projet: Action Intégrée franco-marocaine 212/MA/00
-PARS MI36

-Principales publications ou communications auxquelles ce travail a donné lieu (en se limitant à celles qui se rapportent à ce travail uniquement):

Publications

1) *Global Attractor for A Class of Partial Functional Differential Equations with Infinite Delay.* **"Topics in Functional Differential and Difference Equations", Edited by P. Freitas and T. Faria, Fields Institute Communications, Vol. 29 ,** ***American Mathematical Society, Providence, RI,*** **(March, 2001).** *(En collaboration avec K. Ezzinbi).*
2) *Existence for a Class of Partial Functional Differential Equations with Infinite Delay.* **A paraître au «Journal of Nonlinear Analysis, Theory, Methods and Applications».** *(En collaboration avec M. Adimy et K. Ezzinbi).*
3) *Local Existence and Stability for Some Partial Functional Differential Equations with Infinite Delay.* **A paraître au «Journal of Nonlinear Analysis, Theory, Methods and Applications».** *(En collaboration avec M. Adimy et K. Ezzinbi).*
4) *Existence of Periodic Solutions for Some Partial Functional Differential Equations with Infinite Delay.* **«Journal of Mathematical Analysis and Applications», Vol. 256, N. 1, 257-280, (April, 2001)**. *(En collaboration avec R. Benkhalti et K. Ezzinbi).*

Articles en cours de rédaction

1) *Existence and Stability for Some Partial Neutral Functional Differential Equations with Infinite Delay. (En collaboration avec M. Adimy et K. Ezzinbi).*
2) *Critical Spectrum and Asymptotic Behavior for Linear Partial Functional Differential Equations with Infinite Delay. (En collaboration avec K. Ezzinbi).*

Communications

1) *Sur les équations différentielles semi-linéaires de type retard infini.* **A «CIMPA Summer School on Evolution Equations and Applications», Ouagadougou, Burkina Faso, 13-31 Juillet, 1998.**
2) *Local Existence and Stability for a Class of Partial Functional Differential Equations with Infinite Delay.* **Au 4ièmes JAUCA, Marrakech, 27-29 Avril, 1999. La partie:** *«Local Existence and Semiflow for Some Partial Functional Differential Equations with Infinite Delay»* **Soumise au numéro spécial du Journal de la Société de Mathématiques du Maroc consacré au 4ièmes JAUCA**. *(En collaboration avec K. Ezzinbi).*

3) *Existence de Solutions pour une Classe d'Equations aux Dérivées Partielles à Retard.* **Au Colloque International sur les équations aux dérivées partielles, Faculté des Sciences Dhar Mahraz, Fès, 4-8 Mai, 1999**.
4) *Semiflow Generated by a Class of Partial Functional Differential Equations with Infinite Delay.* **A «International Conference on Functional Differential and Difference Equations» Instituto Superior Técnico, Lisboa-Portugal, 26-30 Juillet, 1999.**
5) *A Massera Type Criterion for Linear Partial Functional Differential Equations with Infinite Delay.* **Au 6ièmes Journées d'Analyse Numérique et Optimisation (JANO6) Facultés de Casablanca, 8-10 Mars, 2000. Paru aux actes de JANO6.**
6) *Critical Spectrum and Asymptotic Behavior for Linear Partial Functional Differential Equations with Infinite Delay.* **A «2nd European-Maghreb Workshop on Semigroups Theory, Evolution Equations and Applications», L'Aquila-Italie, 25-30 Juin, 2000.**
7) *On Existence of Periodic Solutions for Some Partial Functional Differential Equations with Infinite Delay*. **A CIMASI'2000, Ecole Hassania des Travaux Publics, Casablanca, 23-25 Octobre, 2000**.

Content

[1] This chapter is based on a paper in collaboration with M. Adimy and K. Ezzinbi. The paper will appear in Journal of Nonlinear Analysis, Theory, Methods and Applications, (2001).

[2] This chapter is based on two papers. The first one is in collaboration with M. Adimy and K. Ezzinbi and it will appear in Journal of Nonlinear Analysis, Theory, Methods and Applications, (2001). The second one is in collaboration with K. Ezzinbi and it will appear in Fields Institue Communications Series, (2001).

[3] This chapter is based on an ongoing work in collaboration with M. Adimy and K. Ezzinbi.

[4] This chapter is based on a paper in collaboration with R. Benkhalti and K. Ezzinbi. The paper will appear in Journal of Mathematical Analysis and Applications, (2001).

PRESENTATION GENERALE

0.1 Introduction

Le travail présenté dans cette thèse se situe dans le cadre des équations différentielles à retard infini en dimension infinie. Nous nous intéressons plus exactement aux équations de type:

$$\begin{cases} \dfrac{d}{dt}x(t) = Ax(t) + F(t, x_t), \quad t \geq 0, \\ x_0 = \varphi \in \mathcal{B}, \end{cases} \tag{0.1}$$

où $A : D(A) \subset E \to E$ est un opérateur linéaire fermé sur un espace de Banach $(E, |.|)$; pour tout $t \geq 0$, la fonction $x_t \in \mathcal{B}$ est définie par:

$$x_t(\theta) = x(t+\theta), \text{ pour } -\infty < \theta \leq 0.$$

$\mathcal{B}$ est l'espace de phase constitué de fonctions définies de $]-\infty, 0]$ à valeurs dans E, vérifiant certains axiomes (voir Chapitre 1). $F : I\!R^+ \times \mathcal{B} \to E$ est une fonction continue.
Le long de cette thèse, nous supposons que $\overline{D(A)} \neq E$ et que la résolvante de A vérifie la condition de Hille-Yosida: il existe $\omega \in I\!R$ tel que l'ensemble résolvant $\rho(A)$ de A contient $]\omega, +\infty[$ et que

$$\sup_{n\geq 0,\ \lambda>\omega} \left\{ (\lambda-\omega)^n \left\| (\lambda I - A)^{-n} \right\| \right\} < +\infty, \tag{0.2}$$

Notons que dans le cas où $\overline{D(A)} = E$, le théorème de Hille-Yosida affirme que A engendre un C_0 semi-groupe $(T(t))_{t\geq 0}$ sur l'espace E. Une solution (faible) de l'équation (0.1) est donnée par la formule de variation de la constante suivante:

$$x(t,\varphi) = \begin{cases} T(t)\varphi(0) + \displaystyle\int_0^t T(t-s)F(s, x_s)ds, \quad \text{pour } t \geq 0, \\ \varphi(t), \quad \text{pour } -\infty < t \leq 0. \end{cases} \tag{0.3}$$

L'étude de l'équation (0.1) a fait l'objet de plusieurs travaux. Nous citons essentiellement Hale et Kato [55]; Arino, Burton et Haddock [20]; Murakami [84]; Hino, Murakami et Naito [66] et plus récemment, Henriquez ([58]-[60]); Hino, Murakami et Yoshizawa [67]; Liu et Xu [77]; Naito, Murakami et Shin [91]; Shin [113]; Shin et Naito [114], et Shin, Naito et Minh [115]. On cite aussi Milota et Petzeltova [82], [83], [98]-[100] et [101] dans le cas où A

engendre un semi groupe analytique dans E et Ruan et Wu [103]; Ruess [104]-[106]; Ruess et Summers [107], [108]; Ruess, Summers et William [109], et Kartsatos et Parrott [69] dans des cas plus géméraus. La liste est loin d'être exhaustive.

Dans ce travail, nous traitons l'existence de solutions faibles ou intégrales et leurs régularités. Nous utilisons la théorie des semi-groupes intégrés introduite par Arendt et al.. Nous entendons par solution faible ou intégrale toute fonction $x : (-\infty, T] \rightarrow E$, $T > 0$, satisfaisant l'équation suivante:

$$x(t, \varphi) = \begin{cases} S'(t)\varphi(0) + \dfrac{d}{dt}\displaystyle\int_0^t S(t-s)F(s, x_s)ds, & \text{pour } t \in [0, T], \\ \varphi(t), \quad \text{pour } -\infty < t \leq 0. \end{cases} \tag{0.4}$$

où $(S(t))_{t\geq 0}$ est le semi-groupe intégré engendré par l'opérateur A.

Dans ce travail, nous présentons des résultats sur les aspects quantitatif et qualitatif pour les équations différentielles à retard infini et de type neutre en dimension infinie. Nous étudions l'existence, l'unicité, la stabilité et la régularité. En suite, nous nous intéressons à l'existence de solutions périodiques, lorsque F est périodique en t.

0.2 Historique et modèles

La modélisation mathématique de certains problèmes naturels conduit généralement à des modèles qui sont continus ou discrets. Dans les modèles continus, on suppose que l'évolution au cours du temps se fait de manière continue. Ils sont présentés par des équations différentielles, des équations aux dérivées partielles ou par des équations intégrales.

Les équations différentielles à retard surviennent dans certains modèles dont l'état à un instant donné, est une fonction de son passé. On peut les rencontrer dans plusieurs domaines d'applications, notamment en économie, physique, médecine, biologie et écologie,.... En effet, dans certains phénomènes, on s'est aperçu que la connaissance de la solution en un point ne suffit pas pour décrire l'évolution sur un intervalle de temps donné. Des retards surgissent à cause du temps nécessaire pour que le système réponde à une certaine évolution, ou parce qu'un certain seuil doit être atteint avant que le système ne soit activé. La signification du retard dans un tel ou tel modèle peut être différente : durée de gestion, période d'incubation d'une maladie contagieuse, temps d'accumulation, temps nécessaire pour la maturation des cellules ou la transformation d'un type de cellules en un autre,....

Les problèmes démographiques ont été les premiers grands incitateurs à l'introduction des retards dans les modèles. Indiquons brièvement les facteurs qui ont conduit à ce type d'équations. Au début, Malthus a présenté le modèle suivant:

$$\frac{d}{dt}N(t) = bN(t), \tag{0.5}$$

où $N(t)$ est le nombre d'individus à l'instant t et b est le taux de fécondité.

On s'aperçoit que N évolue exponentiellement par rapport à t, et par conséquent, ce modèle ne reflète pas l'évolution exacte de l'espèce, d'où la nécessité d'introduire d'autres modèles plus réalistes. Les modèles à retard ont pour objet de résoudre ce problème. Le modèle "proie-prédateur" de Volterra est constitué de deux populations l'une prédateur se nourrissant de l'autre, la proie. Volterra a supposé que la croissance des prédateurs en contact avec la proie n'est pas instantannée, dûe par exemple à une période de gestation. Parmi les modèles les plus classiques, on trouve le système à retard suivant:

$$\begin{cases} \frac{d}{dt}N_1(t) = (b_1 - a_1 N_2(t))N_1(t), \\ \frac{d}{dt}N_2(t) = \left[-b_2 + a_2 N_1(t) + \int_{t-r}^{t} k(\theta - t)N_1(\theta)d\theta\right] N_2(t), \end{cases} \tag{0.6}$$

où k est une fonction qui décrit la manière dont le gain du prédateur à chaque instant t dépend de l'abondance de la population des proies N_1 dans un intervalle de temps passé $[t-r, t]$. On distingue deux cas: modèle avec retard fini $(0 < r < +\infty)$ et modèle avec retard infini $(r = +\infty)$. Le modèle peut s'écrire alors sous la forme:

$$\frac{d}{dt}N(t) = F(t, N_t), \quad t \geq 0, \tag{0.7}$$

avec la notation $N_t(\theta) = N(t+\theta)$ pour chaque $-\infty < \theta \leq 0$ si $r = +\infty$ ou pour chaque $-r \leq \theta \leq 0$ si $0 < r < +\infty$. La fonction $F : \mathbb{R}^+ \times X \to E$, où X est un espace vectoriel approprié de fonctions allant de $]-\infty, 0]$ à un espace de Banach E si $r = +\infty$ ou de $[-r, 0]$ à E si $0 < r < +\infty$. Dans l'exemple (0.6), la fonction F est défine par

$$F(t, \phi) = \begin{pmatrix} (b_1 - a_1\phi_2(0))\phi_1(0) \\ \left[-b_2 + a_2\phi_1(0) + \int_{-r}^{0} k(\theta)\phi_1(\theta)d\theta\right] \phi_2(0) \end{pmatrix}, \tag{0.8}$$

pour $E = \mathbb{R}^2$ et $\phi = (\phi_1, \phi_2)$.

La modélisation mathématique de certains problèmes en dynamique de populations a conduit à de nouveaux modèles qui sont représentés par des équations aux dérivées partièlles à retard. Dans ces modèles, on tient compte aussi de l'évolution par rapport à d'autres paramètres du système, par exemple l'espace, l'âge, ... Dans la suite, nous présentons quelques modèles qui ont été obtenus dans la littérature et qui peuvent être transformés en des équations de type (0.1) avec retard fini ou infini. On se contente de donner trois exemples de modèles de dynamique de populations structurées. Pour d'autres exemples, nous renvoyons le lecteur aux livres de Wu [124] et Webb [122].

0.2.1 Modèle de prolifération cellulaire

C'est un modèle de production du sang proposé par Rey et Mackey en 1993 et étudié par Dyson, Villella-Bressan et Webb en 1995. Ce modèle décrit la production des souches prolifératives et le précurseur des cellules dans la moelle oseuse:

$$\begin{cases} \dfrac{\partial}{\partial t}u(x,t) + \dfrac{\partial}{\partial x}(xu(x,t)) = \mu u(\alpha x, t-r)(1-u(\alpha x, t-r)), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 < x < 1,\ \ t > 0, \\ u(x,\theta) = \phi(x,t),\ 0 \le x \le 1,\ \ -r \le \theta \le 0, \end{cases} \tag{0.9}$$

où $u(x,t)$ est la densité de population des cellules dépendante de la maturité x, ou l'âge biologique, et du temps t et μ, α et r des paramètres satisfaisant $\mu > 0$, $0 < \alpha < 1$ et $r > 0$. La variable de maturité x est à valeurs dans $[0,1]$ et peut être reliée a l'hemoglobine qui se trouve entre les cellules individuelles. Dans le terme de transport $\dfrac{\partial}{\partial x}(xu(x,t))$, on suppose que toutes les cellules ont un même taux de maturation x. Le retard r et le facteur de maturité α surviennent lorsqu'on suppose que toutes les cellules se subdivisent exactement en un même âge. La dépendance logistique non linéaire de la densité de population, dans le terme $\mu u(\alpha x, t-r)(1-u(\alpha x, t-r))$, signifie que le processus de division des cellules n'est pas modélisé directement, mais il y'a une prodution de nouvelles cellules de toutes les valeurs de maturité.

0.2.2 Modèle de proie-prédateur avec diffusion dans l'espace

Pour l'étude d'une population proie-prédateur, Cohen, Hagan et Simpson [34] ont proposé le modèle suivant:

$$\begin{aligned} \frac{\partial}{\partial t}u(x,t) = \alpha\Delta u(x,t) + h&\left(\int_0^{+\infty} K(s)u(x,t-s)ds\right) \\ -m&\left(\int_0^{+\infty} H(s)u(x,t-s)ds\right), \end{aligned} \tag{0.10}$$

où $u(x,t)$ est la population de la proie au temps t et à la position x. K et H sont les fonctions poids des effets héréditaires. Le terme $h\left(\int_0^{+\infty} K(s)u(x,t-s)ds\right)$ décrit les processus de croissance et de décroissance de la proie et $m\left(\int_0^{+\infty} H(s)u(x,t-s)ds\right)$ est la consommation de la population de la proie par le prédateur.

0.2.3 Modèle de dynamique d'une population distribuée

Il s'agit du modèle à retard infini suivant:

$$\begin{cases} \dfrac{\partial}{\partial t}N(x,t) = d\dfrac{\partial^2}{\partial x^2}N(x,t) + cN(x,t)\left[1 - \displaystyle\int_{-\infty}^{0} G(\theta)N(x,t+\theta)d\theta\right], \\ \qquad\qquad\qquad\qquad\qquad 0 < x < \pi, \;\; t > 0, \\ \dfrac{\partial}{\partial x}N(0,t) = \dfrac{\partial}{\partial x}N(\pi,t) = 0, \;\; t > 0, \\ N(x,\theta) = \phi(\theta)(x), \;\; 0 \leq x \leq \pi, \;\; -\infty < \theta \leq 0, \end{cases} \tag{0.11}$$

qui décrit les dynamiques d'une population distribuée d'une seule espèce. $N(x,t)$ représente la taille de la population totale à la position x et à l'instant t, les deux constantes d et c sont positives mesurant, respectivement, le taux de diffusion et la croissance intrinsèque et G : $]-\infty, 0] \to I\!R^+$ est une fonction intégrable vérifiant $\int_{-\infty}^{0} G(\theta)d\theta = 1$. La fonction ϕ est un élément d'un espace vectoriel approprié de fonctions allant de $]-\infty, 0]$ vers $E := L^2([0,\pi])$. Le modèle a été étudié par plusieurs auteurs, à savoir: Ruan et Wu [103], Bonilla et Linan [27], et Britton [30].

0.3 Equations différentielles à retard fini en dimension infinie

Dans le cas du retard fini, l'équation (0.1) s'écrit sous la forme:

$$\begin{cases} \dfrac{d}{dt}x(t) = Ax(t) + F(t,x_t), \quad t \geq 0, \\ x_0 = \varphi \in \mathcal{C}, \end{cases} \tag{0.12}$$

où x_t est définie par $x_t(\theta) = x(t+\theta)$, pour $-r \leq \theta \leq 0$, $r > 0$, et $\mathcal{C}$ est l'espace des fonctions continues de $[-r, 0]$ à valeurs dans E muni de la topologie de la convergence uniforme. Lorsque A est générateur infinitésimal d'un C_0-semi-groupe $(T(t))_{t\geq 0}$ dans E, l'équation (0.12) a fait l'objet de plusieurs travaux. Dans [118], Travis et Webb ont formulé tous les aspects d'existence et stabilité pour l'équation (0.12). Dans le cas où F est autonome et globalement Lipschitzienne, ils ont démontré que la solution:

$$x(t,\varphi) = \begin{cases} T(t)\varphi(0) + \displaystyle\int_0^t T(t-s)F(x_s)ds, \quad \text{pour } t \geq 0, \\ \varphi(t), \quad \text{pour } -r \leq t \leq 0, \end{cases} \tag{0.13}$$

définit un semi-groupe non linéaire $(U(t))_{t\geq 0}$ sur $\mathcal{C}$. Lorsque le semi-groupe $(T(t))_{t\geq 0}$ est compact, ils ont démontré que le semi-groupe solution $U(t)$ est compact pour $t > r$. Cette dernière propriété a permis dans le cas linéaire d'étudier la stabilité en terme d'une équation caractéristique qui caractérise le spectre ponctuel du générateur infinitésimal de $(U(t))_{t\geq 0}$. Cette hypothèse de compacité a permis aussi de formuler le théorème de variété centre pour l'équation (0.12), à savoir que l'espace $\mathcal{C}$ s'écrit comme somme directe de trois parties stable, centre et instable. Dans [23] et [24], Arino et Sanchez ont traité l'équation (0.12) lorsque $A = 0$, et ont donné une formule de variation de la constante. Dans [80] et [81], Memory a étudié le problème d'existence d'une variété stable et instable pour l'équation (0.12). Il a aussi donné une formule de variation de la constante. Dans [76], Lin, So et Wu ont étudié l'existence d'une variété centre pour l'équation (0.12). Dans le cas où l'espace instable est réduit à $\{0\}$, ils ont montré que la variété centre obtenue est exponentiellement attractive. D'autres résultats de base se trouvent dans [119], [120] et [121]. Par la suite, d'autres travaux ont été faits par plusieurs auteurs: Kunisch et Schappacher ([72], [73]), Grabosch et Moustakas [46] et Parrott [95] et [96]. Pour plus de détails sur ce sujet, nous référons le lecteur au livre de Hale et Lunel [56] et à celui de Wu [124]. Récemment, dans [1], [3], [7] et [45], Adimy et Ezzinbi ont repris l'étude de l'équation (0.12) dans le cas où l'opérateur A est à domaine non dense et vérifie la condition de Hille-Yosida (0.2).

0.4 Equations différentielles à retard infini en dimension infinie

La théorie des équations différentielles à retard infini en dimension finie a été initiée par Hale et Kato dans l'article [55]. Les auteurs ont formulé des axiomes sur l'espace de phase $\mathcal{B}$, pour pouvoir étudier les problèmes quantitatifs et qualitatifs. Cette approche a permis par la suite à plusieurs auteurs de développer une théorie générale pour les équations différentielles à retard infini en dimension finie et infinie. Nous citons, entre autres, Arino, Burton et Haddock [20]; Henriquez [58], [58] et [60]; Ruan et Wu [103]; Hino, Murakami et Yoshizawa [67]; Naito, Shin et Murakami [91] et [92]; Shin et Naito [114]; et Shin, Naito et Minh [115].

La théorie des équations différentielles à retard infini a connu un développement considérable. Plusieurs résultats ont été obtenus dans le cas où A engendre un C_0-semi-groupe dans E. Dans ce qui suit, nous décrivons les résultats obtenus dans les travaux cités ci-dessus.

Dans [20], Arino et al. ont étudié l'équation (0.1) avec $A = 0$. Les auteurs ont démontré

moyennant le théorème du point fixe de Horn, l'existence de solutions périodiques lorsqu'il y a existence de solutions ultimement bornées.
Dans [58]-[60], Henriquez a traité le problème d'existence de solutions pour l'équation (0.1) et leur régularité. Dans [58] et [59], l'auteur a étudié aussi l'existence de solutions périodiques dans le cas où F est périodique en t.
Dans [103], Ruan et Wu ont montré que sous certaines conditions sur la fonction F et sur l'espace de phase, l'opérateur solution est une α-contraction.
Dans [114], Shin et Naito ont étudié le problème d'existence de solutions périodiques dans le cas où F est linéaire par rapport à la deuxième variable. Moyennant des hypothèses de compacité, ils ont montré que l'opérateur de Poincaré associé a l'équation est une α-contraction. Ce qui a permis grâce à un théorème du point fixe de Chow et Hale [33], de démontrer que l'opérateur de Poincaré admet un point fixe qui est la donnée initiale d'une solution périodique.
Dans [91], [92] et [115], Shin, Naito, Minh et Murakami ont donné des propriétés de stabilité pour le semi-groupe solution d'une équation différentielle à retard infini en dimension infinie. Ils ont aussi caractérisé le générateur infinitésimal du semi-groupe solution.
Dans [67], Hino, Murakami et Yoshizawa ont traité le problème d'existence de solutions presque périodiques dans le cas où F est linéaire par rapport à la deuxième variable.

0.5 Equations différentielles à retard de type neutre en dimension infinie

Dans [53] et [54], Hale a proposé l'étude d'un modèle de circuits éléctriques. Ce modèle est présenté par une équation aux dérivées partielles à retard fini sous la forme:

$$\begin{cases} \dfrac{\partial}{\partial t} D v_t = k \dfrac{\partial^2}{\partial x^2} D v_t + F(v_t), \quad t \geq 0, \\ v_0 = \varphi \in C, \end{cases} \tag{0.14}$$

avec k une constante positive, $C := C\left([-r, 0]; H^1(S^1)\right)$ est l'espace des fonctions continues sur $[-r, 0]$ à valeurs dans l'espace de Sobolev $H^1(S^1)$, S^1 est la boule unité, $F : C \to H^1(S^1)$ est une fonction assez régulière et $D\phi(s) := \phi(\theta)(0) - \displaystyle\int_{-r}^{0} [d\eta(\theta)]\, \phi(\theta)(s)$ pour $s \in S^1$ et $\phi \in C$, où η est à variation bornée et atomique en 0, c'est à dire, il existe une fonction croissante $\delta : [0,r] \to [0,+\infty)$ telle que $\delta(0) = 0$ et

$$\left| \int_{-s}^{0} [d\eta(\theta)]\, \psi(\theta) \right| \leq \delta(s) \sup_{-r \leq \theta \leq 0} |\psi(\theta)|\,,\ \psi \in C.$$

Hale a initié l'étude de l'existence, la stabilité, l'attractivité et la bifurcation pour les équations de type (0.14). Il a considéré l'opérateur de Laplace $A = k\dfrac{\partial^2}{\partial x^2}$ avec domaine $H^2(S^1)$, ce qui donne un générateur infinitésimal d'un C_0 semi-groupe sur $E = H^1(S^1)$.

Dans leur description d'un circuit éléctrique formé de plusieurs oscillateurs identiques, connectés entre eux par une résistance et formant une boucle fermée, Wu et Xia [125] et [126] ont aboutit à un système hyperbolique qui est équivalent à une équation différentielle de type neutre à retard fini. Ils ont considéré des équations de la forme:

$$\frac{\partial}{\partial t}\left[x(\xi,t) - qx(\xi,t-h)\right] = k\frac{\partial^2}{\partial \xi^2}\left[x(\xi,t) - qx(\xi,t-h)\right] + f\left(x_t(\xi,.)\right), \quad \xi \in S^1,\ t \geq 0, \tag{0.15}$$

où $x_t(\xi,\theta) = x(\xi, t+\theta)$, $-h \leq \theta \leq 0$, $t \geq 0$, $\xi \in S^1$, k est une constante positive, f est une fonction continue et $0 \leq q < 1$.

Le livre de Wu [124] contient une analyse détaillée des résultats obtenus dans les articles [53], [54], [125] et [126]. Puisque $H^1(S^1) \subset C(S^1)$, dans [124], l'auteur a également considéré l'opérateur de Laplace $A = k\dfrac{\partial^2}{\partial x^2}$ avec domaine $C^2(S^1)$, qui est générateur infinitésimal d'un C_0 semi-groupe sur $E = C(S^1)$. Ce qui a permis d'obtenir des résultats analogues dans un cadre plus général : $C := C\left([-r,0]\,;C(S^1)\right)$.

Récemment, dans [10], [11] et [12], Adimy et Ezzinbi ont considéré la forme générale suivante:

$$\begin{cases} \dfrac{d}{dt}\left[x(t) - G(t,x_t)\right] = A\left[x(t) - G(t,x_t)\right] + F(t,x_t), \quad t \geq 0, \\ x_0 = \varphi \in \mathcal{C}, \end{cases} \tag{0.16}$$

avec G et F deux fonctions continues sur $[0,+\infty[\, \times \mathcal{C} \to E$. Les auteurs ont développé des résultats sur l'existence, la régularité et la stabilité dans le cas où l'opérateur A est à domaine non dense et vérifie la condition de Hille-Yosida (0.2).

Notre contribution à l'étude des équations de type neutre à retard infini concerne la reproduction de quelques résultats concernant l'étude de l'équation (0.16) dans le cas où le retard est infini. Plus précisément, nous nous intéressons à l'équation suivante:

$$\begin{cases} \dfrac{d}{dt}\left[x(t) - G(t,x_t)\right] = A\left[x(t) - G(t,x_t)\right] + F(t,x_t), \quad t \geq 0, \\ x(\theta) = \varphi(\theta), \ \ -\infty < \theta \leq 0, \end{cases} \tag{0.17}$$

avec $\varphi \in \mathcal{B}$, la fonction $G : [0, +\infty[\times \mathcal{B} \to E$ est continue et les mêmes notations que l'équation (0.1).
Signalons que Hernandez et Henriquez [61], [62] et [63] ont étudié l'équation (0.17) dans le cas où A est générateur d'un semigroupe analytique dans E et G est à valeurs dans le domaine d'une puissance fractionnaire de $-A$, $(D((-A)^{-\alpha}), 0 < \alpha < 1)$.

0.6 Description de la thèse

Dans la suite, nous donnons un résumé des résultats développés dans chaque chapitre.

Dans le chapitre 1, nous avons présenté quelques résultats d'analyse fonctionnelle qui seront utilisés le long de ce travail. La première section est une présentation axiomatique de la théorie fondamentale de l'espace de phase pour l'étude des équations différentielles à retard infini. Dans la deuxième section, nous avons donné des résultats sur la théorie des semi-groupes intégrés.

Dans le chapitre 2, nous avons étudié l'existence et la régularité des solutions pour l'équation (0.1), dans le cas où F est continue et Lipschitzienne par rapport à la deuxième variable.

Dans le chapitre 3, nous nous intéressons à l'aspect semi-groupe de la solution. Nous avons étudié l'existence et la régularité des solutions dans le cas où F est localement Lipschitzienne. Dans le cas où l'existence globale est vérifiée, nous avons démontré que la solution définit un semi-groupe qui vérifie la propriété de translation. Ceci nous a permis d'étudier la stabilité d'un point d'équilibre et de démontrer l'existence d'un attracteur global.

Dans le chapitre 4, nous avons donné des conditions suffisantes d'existence et de régularité pour l'équation (0.17). Dans la cas autonome, nous avons étudié la stabilité d'un point d'équilibre.

Dans le chapitre 5, nous avons traité l'existence de solutions périodiques pour l'équation (0.1) dans le cas où F est périodique en t. Dans le cas où F est linéaire par rapport à la deuxième variable, nous avons démontré que l'existence d'une solution bornée entraine l'existence d'une solution périodique.

Content

[1] This chapter is based on a paper in collaboration with M. Adimy and K. Ezzinbi. The paper will appear in Journal of Nonlinear Analysis, Theory, Methods and Applications, (2001).

[2] This chapter is based on two papers. The first one is in collaboration with M. Adimy and K. Ezzinbi and it will appear in Journal of Nonlinear Analysis, Theory, Methods and Applications, (2001). The second one is in collaboration with K. Ezzinbi and it will appear in Fields Institue Communications Series, (2001).

[3] This chapter is based on an ongoing work in collaboration with M. Adimy and K. Ezzinbi.

[4] This chapter is based on a paper in collaboration with R. Benkhalti and K. Ezzinbi. The paper will appear in Journal of Mathematical Analysis and Applications, (2001).

PRESENTATION GENERALE

0.1 Introduction

Le travail présenté dans cette thèse se situe dans le cadre des équations différentielles à retard infini en dimension infinie. Nous nous intéressons plus exactement aux équations de type:

$$\begin{cases} \dfrac{d}{dt}x(t) = Ax(t) + F(t, x_t), \quad t \geq 0, \\ x_0 = \varphi \in \mathcal{B}, \end{cases} \tag{0.1}$$

où $A : D(A) \subset E \to E$ est un opérateur linéaire fermé sur un espace de Banach $(E, |.|)$; pour tout $t \geq 0$, la fonction $x_t \in \mathcal{B}$ est définie par:

$$x_t(\theta) = x(t + \theta), \text{ pour } -\infty < \theta \leq 0.$$

$\mathcal{B}$ est l'espace de phase constitué de fonctions définies de $]-\infty, 0]$ à valeurs dans E, vérifiant certains axiomes (voir Chapitre 1). $F : I\!R^+ \times \mathcal{B} \to E$ est une fonction continue.
Le long de cette thèse, nous supposons que $\overline{D(A)} \neq E$ et que la résolvante de A vérifie la condition de Hille-Yosida: il existe $\omega \in I\!R$ tel que l'ensemble résolvant $\rho(A)$ de A contient $]\omega, +\infty[$ et que

$$\sup_{n \geq 0,\ \lambda > \omega} \left\{ (\lambda - \omega)^n \left\| (\lambda I - A)^{-n} \right\| \right\} < +\infty, \tag{0.2}$$

Notons que dans le cas où $\overline{D(A)} = E$, le théorème de Hille-Yosida affirme que A engendre un C_0 semi-groupe $(T(t))_{t \geq 0}$ sur l'espace E. Une solution (faible) de l'équation (0.1) est donnée par la formule de variation de la constante suivante:

$$x(t, \varphi) = \begin{cases} T(t)\varphi(0) + \displaystyle\int_0^t T(t - s)F(s, x_s)ds, \quad \text{pour } t \geq 0, \\ \varphi(t), \quad \text{pour } -\infty < t \leq 0. \end{cases} \tag{0.3}$$

L'étude de l'équation (0.1) a fait l'objet de plusieurs travaux. Nous citons essentiellement Hale et Kato [55]; Arino, Burton et Haddock [20]; Murakami [84]; Hino, Murakami et Naito [66] et plus récemment, Henriquez ([58]-[60]); Hino, Murakami et Yoshizawa [67]; Liu et Xu [77]; Naito, Murakami et Shin [91]; Shin [113]; Shin et Naito [114], et Shin, Naito et Minh [115]. On cite aussi Milota et Petzeltova [82], [83], [98]-[100] et [101] dans le cas où A

engendre un semi groupe analytique dans E et Ruan et Wu [103]; Ruess [104]-[106]; Ruess et Summers [107], [108]; Ruess, Summers et William [109], et Kartsatos et Parrott [69] dans des cas plus généraus. La liste est loin d'être exhaustive.

Dans ce travail, nous traitons l'existence de solutions faibles ou intégrales et leurs régularités. Nous utilisons la théorie des semi-groupes intégrés introduite par Arendt et al.. Nous entendons par solution faible ou intégrale toute fonction $x : (-\infty, T] \to E$, $T > 0$, satisfaisant l'équation suivante:

$$x(t,\varphi) = \begin{cases} S'(t)\varphi(0) + \dfrac{d}{dt}\displaystyle\int_0^t S(t-s)F(s,x_s)ds, & \text{pour } t \in [0,T], \\ \varphi(t), \quad \text{pour } -\infty < t \leq 0. \end{cases} \tag{0.4}$$

où $(S(t))_{t\geq 0}$ est le semi-groupe intégré engendré par l'opérateur A.

Dans ce travail, nous présentons des résultats sur les aspects quantitatif et qualitatif pour les équations différentielles à retard infini et de type neutre en dimension infinie. Nous étudions l'existence, l'unicité, la stabilité et la régularité. En suite, nous nous intéressons à l'existence de solutions périodiques, lorsque F est périodique en t.

0.2 Historique et modèles

La modélisation mathématique de certains problèmes naturels conduit généralement à des modèles qui sont continus ou discrets. Dans les modèles continus, on suppose que l'évolution au cours du temps se fait de manière continue. Ils sont présentés par des équations différentielles, des équations aux dérivées partielles ou par des équations intégrales.

Les équations différentielles à retard surviennent dans certains modèles dont l'état à un instant donné, est une fonction de son passé. On peut les rencontrer dans plusieurs domaines d'applications, notamment en économie, physique, médecine, biologie et écologie,.... En effet, dans certains phénomènes, on s'est aperçu que la connaissance de la solution en un point ne suffit pas pour décrire l'évolution sur un intervalle de temps donné. Des retards surgissent à cause du temps nécessaire pour que le système réponde à une certaine évolution, ou parce qu'un certain seuil doit être atteint avant que le système ne soit activé. La signification du retard dans un tel ou tel modèle peut être différente : durée de gestion, période d'incubation d'une maladie contagieuse, temps d'accumulation, temps nécessaire pour la maturation des cellules ou la transformation d'un type de cellules en un autre,....

Les problèmes démographiques ont été les premiers grands incitateurs à l'introduction des retards dans les modèles. Indiquons brièvement les facteurs qui ont conduit à ce type d'équations. Au début, Malthus a présenté le modèle suivant:

$$\frac{d}{dt}N(t) = bN(t), \tag{0.5}$$

où $N(t)$ est le nombre d'individus à l'instant t et b est le taux de fécondité.

On s'aperçoit que N évolue exponentiellement par rapport à t, et par conséquent, ce modèle ne reflète pas l'évolution exacte de l'espèce, d'où la nécessité d'introduire d'autres modèles plus réalistes. Les modèles à retard ont pour objet de résoudre ce problème. Le modèle "proie-prédateur" de Volterra est constitué de deux populations l'une prédateur se nourrissant de l'autre, la proie. Volterra a supposé que la croissance des prédateurs en contact avec la proie n'est pas instantannée, dûe par exemple à une période de gestation. Parmi les modèles les plus classiques, on trouve le système à retard suivant:

$$\begin{cases} \frac{d}{dt}N_1(t) = (b_1 - a_1N_2(t))N_1(t), \\ \frac{d}{dt}N_2(t) = \left[-b_2 + a_2N_1(t) + \int_{t-r}^{t} k(\theta - t)N_1(\theta)d\theta\right] N_2(t), \end{cases} \tag{0.6}$$

où k est une fonction qui décrit la manière dont le gain du prédateur à chaque instant t dépend de l'abondance de la population des proies N_1 dans un intervalle de temps passé $[t-r, t]$. On distingue deux cas: modèle avec retard fini ($0 < r < +\infty$) et modèle avec retard infini ($r = +\infty$). Le modèle peut s'écrire alors sous la forme:

$$\frac{d}{dt}N(t) = F(t, N_t), \quad t \geq 0, \tag{0.7}$$

avec la notation $N_t(\theta) = N(t+\theta)$ pour chaque $-\infty < \theta \leq 0$ si $r = +\infty$ ou pour chaque $-r \leq \theta \leq 0$ si $0 < r < +\infty$. La fonction $F : \mathbb{R}^+ \times X \to E$, où X est un espace vectoriel approprié de fonctions allant de $]-\infty, 0]$ à un espace de Banach E si $r = +\infty$ ou de $[-r, 0]$ à E si $0 < r < +\infty$. Dans l'exemple (0.6), la fonction F est défine par

$$F(t, \phi) = \begin{pmatrix} (b_1 - a_1\phi_2(0))\phi_1(0) \\ \left[-b_2 + a_2\phi_1(0) + \int_{-r}^{0} k(\theta)\phi_1(\theta)d\theta\right] \phi_2(0) \end{pmatrix}, \tag{0.8}$$

pour $E = \mathbb{R}^2$ et $\phi = (\phi_1, \phi_2)$.

La modélisation mathématique de certains problèmes en dynamique de populations a conduit à de nouveaux modèles qui sont représentés par des équations aux dérivées partièlles à retard. Dans ces modèles, on tient compte aussi de l'évolution par rapport à d'autres paramètres du système, par exemple l'espace, l'âge, ... Dans la suite, nous présentons quelques modèles qui ont été obtenus dans la littérature et qui peuvent être transformés en des équations de type (0.1) avec retard fini ou infini. On se contente de donner trois exemples de modèles de dynamique de populations structurées. Pour d'autres exemples, nous renvoyons le lecteur aux livres de Wu [124] et Webb [122].

0.2.1 Modèle de prolifération cellulaire

C'est un modèle de production du sang proposé par Rey et Mackey en 1993 et étudié par Dyson, Villella-Bressan et Webb en 1995. Ce modèle décrit la production des souches prolifératives et le précurseur des cellules dans la moelle oseuse:

$$\begin{cases} \dfrac{\partial}{\partial t}u(x,t) + \dfrac{\partial}{\partial x}(xu(x,t)) = \mu u(\alpha x, t-r)(1-u(\alpha x, t-r)), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 < x < 1, \;\; t > 0, \\ u(x,\theta) = \phi(x,t), \; 0 \le x \le 1, \;\; -r \le \theta \le 0, \end{cases} \tag{0.9}$$

où $u(x,t)$ est la densité de population des cellules dépendante de la maturité x, ou l'âge biologique, et du temps t et μ, α et r des paramètres satisfaisant $\mu > 0$, $0 < \alpha < 1$ et $r > 0$. La variable de maturité x est à valeurs dans $[0,1]$ et peut être reliée a l'hemoglobine qui se trouve entre les cellules individuelles. Dans le terme de transport $\dfrac{\partial}{\partial x}(xu(x,t))$, on suppose que toutes les cellules ont un même taux de maturation x. Le retard r et le facteur de maturité α surviennent lorsqu'on suppose que toutes les cellules se subdivisent exactement en un même âge. La dépendance logistique non linéaire de la densité de population, dans le terme $\mu u(\alpha x, t-r)(1-u(\alpha x, t-r))$, signifie que le processus de division des cellules n'est pas modélisé directement, mais il y'a une prodution de nouvelles cellules de toutes les valeurs de maturité.

0.2.2 Modèle de proie-prédateur avec diffusion dans l'espace

Pour l'étude d'une population proie-prédateur, Cohen, Hagan et Simpson [34] ont proposé le modèle suivant:

$$\begin{aligned} \frac{\partial}{\partial t}u(x,t) = \alpha\Delta u(x,t) + h\left(\int_0^{+\infty} K(s)u(x,t-s)ds\right) \\ -m\left(\int_0^{+\infty} H(s)u(x,t-s)ds\right), \end{aligned} \tag{0.10}$$

où $u(x,t)$ est la population de la proie au temps t et à la position x. K et H sont les fonctions poids des effets héréditaires. Le terme $h\left(\int_0^{+\infty} K(s)u(x,t-s)ds\right)$ décrit les processus de croissance et de décroissance de la proie et $m\left(\int_0^{+\infty} H(s)u(x,t-s)ds\right)$ est la consommation de la population de la proie par le prédateur.

0.2.3 Modèle de dynamique d'une population distribuée

Il s'agit du modèle à retard infini suivant:

$$\begin{cases} \dfrac{\partial}{\partial t}N(x,t) = d\dfrac{\partial^2}{\partial x^2}N(x,t) + cN(x,t)\left[1 - \displaystyle\int_{-\infty}^{0} G(\theta)N(x,t+\theta)d\theta\right], \\ \qquad\qquad\qquad\qquad\qquad 0 < x < \pi, \;\; t > 0, \\ \dfrac{\partial}{\partial x}N(0,t) = \dfrac{\partial}{\partial x}N(\pi,t) = 0, \;\; t > 0, \\ N(x,\theta) = \phi(\theta)(x), \;\; 0 \le x \le \pi, \;\; -\infty < \theta \le 0, \end{cases} \tag{0.11}$$

qui décrit les dynamiques d'une population distribuée d'une seule espèce. $N(x,t)$ représente la taille de la population totale à la position x et à l'instant t, les deux constantes d et c sont positives mesurant, respectivement, le taux de diffusion et la croissance intrinsèque et $G :]-\infty,0] \to I\!R^+$ est une fonction intégrable vérifiant $\int_{-\infty}^{0} G(\theta)d\theta = 1$. La fonction ϕ est un élément d'un espace vectoriel approprié de fonctions allant de $]-\infty,0]$ vers $E := L^2([0,\pi])$. Le modèle a été étudié par plusieurs auteurs, à savoir: Ruan et Wu [103], Bonilla et Linan [27], et Britton [30].

0.3 Equations différentielles à retard fini en dimension infinie

Dans le cas du retard fini, l'équation (0.1) s'écrit sous la forme:

$$\begin{cases} \dfrac{d}{dt}x(t) = Ax(t) + F(t,x_t), \quad t \ge 0, \\ x_0 = \varphi \in \mathcal{C}, \end{cases} \tag{0.12}$$

où x_t est définie par $x_t(\theta) = x(t+\theta)$, pour $-r \le \theta \le 0$, $r > 0$, et $\mathcal{C}$ est l'espace des fonctions continues de $[-r,0]$ à valeurs dans E muni de la topologie de la convergence uniforme. Lorsque A est générateur infinitésimal d'un C_0-semi-groupe $(T(t))_{t\ge0}$ dans E, l'équation (0.12) a fait l'objet de plusieurs travaux. Dans [118], Travis et Webb ont formulé tous les aspects d'existence et stabilité pour l'équation (0.12). Dans le cas où F est autonome et globalement Lipschitzienne, ils ont démontré que la solution:

$$x(t,\varphi) = \begin{cases} T(t)\varphi(0) + \displaystyle\int_0^t T(t-s)F(x_s)ds, \quad \text{pour } t \ge 0, \\ \varphi(t), \quad \text{pour } -r \le t \le 0, \end{cases} \tag{0.13}$$

définit un semi-groupe non linéaire $(U(t))_{t\geq 0}$ sur $\mathcal{C}$. Lorsque le semi-groupe $(T(t))_{t\geq 0}$ est compact, ils ont démontré que le semi-groupe solution $U(t)$ est compact pour $t > r$. Cette dernière propriété a permis dans le cas linéaire d'étudier la stabilité en terme d'une équation caractéristique qui caractérise le spectre ponctuel du générateur infinitésimal de $(U(t))_{t\geq 0}$. Cette hypothèse de compacité a permis aussi de formuler le théorème de variété centre pour l'équation (0.12), à savoir que l'espace $\mathcal{C}$ s'écrit comme somme directe de trois parties stable, centre et instable. Dans [23] et [24], Arino et Sanchez ont traité l'équation (0.12) lorsque $A = 0$, et ont donné une formule de variation de la constante. Dans [80] et [81], Memory a étudié le problème d'existence d'une variété stable et instable pour l'équation (0.12). Il a aussi donné une formule de variation de la constante. Dans [76], Lin, So et Wu ont étudié l'existence d'une variété centre pour l'équation (0.12). Dans le cas où l'espace instable est réduit à $\{0\}$, ils ont montré que la variété centre obtenue est exponentiellement attractive. D'autres résultats de base se trouvent dans [119], [120] et [121]. Par la suite, d'autres travaux ont été faits par plusieurs auteurs: Kunisch et Schappacher ([72], [73]), Grabosch et Moustakas [46] et Parrott [95] et [96]. Pour plus de détails sur ce sujet, nous référons le lecteur au livre de Hale et Lunel [56] et à celui de Wu [124]. Récemment, dans [1], [3], [7] et [45], Adimy et Ezzinbi ont repris l'étude de l'équation (0.12) dans le cas où l'opérateur A est à domaine non dense et vérifie la condition de Hille-Yosida (0.2).

0.4 Equations différentielles à retard infini en dimension infinie

La théorie des équations différentielles à retard infini en dimension finie a été initiée par Hale et Kato dans l'article [55]. Les auteurs ont formulé des axiomes sur l'espace de phase $\mathcal{B}$, pour pouvoir étudier les problèmes quantitatifs et qualitatifs. Cette approche a permis par la suite à plusieurs auteurs de développer une théorie générale pour les équations différentielles à retard infini en dimension finie et infinie. Nous citons, entre autres, Arino, Burton et Haddock [20]; Henriquez [58], [58] et [60]; Ruan et Wu [103]; Hino, Murakami et Yoshizawa [67]; Naito, Shin et Murakami [91] et [92]; Shin et Naito [114]; et Shin, Naito et Minh [115].

La théorie des équations différentielles à retard infini a connu un développement considérable. Plusieurs résultats ont été obtenus dans le cas où A engendre un C_0-semi-groupe dans E. Dans ce qui suit, nous décrivons les résultats obtenus dans les travaux cités ci-dessus.

Dans [20], Arino et al. ont étudié l'équation (0.1) avec $A = 0$. Les auteurs ont démontré

moyennant le théorème du point fixe de Horn, l'existence de solutions périodiques lorsqu'il y a existence de solutions ultimement bornées.
Dans [58]-[60], Henriquez a traité le problème d'existence de solutions pour l'équation (0.1) et leur régularité. Dans [58] et [59], l'auteur a étudié aussi l'existence de solutions périodiques dans le cas où F est périodique en t.
Dans [103], Ruan et Wu ont montré que sous certaines conditions sur la fonction F et sur l'espace de phase, l'opérateur solution est une α-contraction.
Dans [114], Shin et Naito ont étudié le problème d'existence de solutions périodiques dans le cas où F est linéaire par rapport à la deuxième variable. Moyennant des hypothèses de compacité, ils ont montré que l'opérateur de Poincaré associé a l'équation est une α-contraction. Ce qui a permis grâce à un théorème du point fixe de Chow et Hale [33], de démontrer que l'opérateur de Poincaré admet un point fixe qui est la donnée initiale d'une solution périodique.
Dans [91], [92] et [115], Shin, Naito, Minh et Murakami ont donné des propriétés de stabilité pour le semi-groupe solution d'une équation différentielle à retard infini en dimension infinie. Ils ont aussi caractérisé le générateur infinitésimal du semi-groupe solution.
Dans [67], Hino, Murakami et Yoshizawa ont traité le problème d'existence de solutions presque périodiques dans le cas où F est linéaire par rapport à la deuxième variable.

0.5 Equations différentielles à retard de type neutre en dimension infinie

Dans [53] et [54], Hale a proposé l'étude d'un modèle de circuits éléctriques. Ce modèle est présenté par une équation aux dérivées partielles à retard fini sous la forme:

$$\begin{cases} \dfrac{\partial}{\partial t} D v_t = k \dfrac{\partial^2}{\partial x^2} D v_t + F(v_t), \quad t \geq 0, \\ v_0 = \varphi \in C, \end{cases} \tag{0.14}$$

avec k une constante positive, $C := C\left([-r, 0]; H^1(S^1)\right)$ est l'espace des fonctions continues sur $[-r, 0]$ à valeurs dans l'espace de Sobolev $H^1(S^1)$, S^1 est la boule unité, $F : C \to H^1(S^1)$ est une fonction assez régulière et $D\phi(s) := \phi(\theta)(0) - \displaystyle\int_{-r}^{0} [d\eta(\theta)]\, \phi(\theta)(s)$ pour $s \in S^1$ et $\phi \in C$, où η est à variation bornée et atomique en 0, c'est à dire, il existe une fonction croissante $\delta : [0,r] \to [0,+\infty)$ telle que $\delta(0)=0$ et

$$\left| \int_{-s}^{0} [d\eta(\theta)]\,\psi(\theta) \right| \le \delta(s) \sup_{-r\le\theta\le 0} |\psi(\theta)|\,,\ \ \psi \in C.$$

Hale a initié l'étude de l'existence, la stabilité, l'attractivité et la bifurcation pour les équations de type (0.14). Il a considéré l'opérateur de Laplace $A = k\dfrac{\partial^2}{\partial x^2}$ avec domaine $H^2(S^1)$, ce qui donne un générateur infinitésimal d'un C_0 semi-groupe sur $E = H^1(S^1)$.

Dans leur description d'un circuit éléctrique formé de plusieurs oscillateurs identiques, connectés entre eux par une résistance et formant une boucle fermée, Wu et Xia [125] et [126] ont aboutit à un système hyperbolique qui est équivalent à une équation différentielle de type neutre à retard fini. Ils ont considéré des équations de la forme:

$$\frac{\partial}{\partial t}\left[x(\xi,t) - qx(\xi,t-h)\right] = k\frac{\partial^2}{\partial \xi^2}\left[x(\xi,t) - qx(\xi,t-h)\right] + f\left(x_t(\xi,.)\right),\quad \xi \in S^1,\ t \ge 0, \tag{0.15}$$

où $x_t(\xi,\theta) = x(\xi,t+\theta)$, $-h \le \theta \le 0$, $t \ge 0$, $\xi \in S^1$, k est une constante positive, f est une fonction continue et $0 \le q < 1$.

Le livre de Wu [124] contient une analyse détaillée des résultats obtenus dans les articles [53], [54], [125] et [126]. Puisque $H^1(S^1) \subset C(S^1)$, dans [124], l'auteur a également considéré l'opérateur de Laplace $A = k\dfrac{\partial^2}{\partial x^2}$ avec domaine $C^2(S^1)$, qui est générateur infinitésimal d'un C_0 semi-groupe sur $E = C(S^1)$. Ce qui a permis d'obtenir des résultats analogues dans un cadre plus général : $C := C\left([-r,0]\,;C(S^1)\right)$.

Récemment, dans [10], [11] et [12], Adimy et Ezzinbi ont considéré la forme générale suivante:

$$\begin{cases} \dfrac{d}{dt}\left[x(t) - G(t,x_t)\right] = A\left[x(t) - G(t,x_t)\right] + F(t,x_t), \quad t \ge 0, \\ x_0 = \varphi \in \mathcal{C}, \end{cases} \tag{0.16}$$

avec G et F deux fonctions continues sur $[0,+\infty[\, \times\, \mathcal{C} \to E$. Les auteurs ont développé des résultats sur l'existence, la régularité et la stabilité dans le cas où l'opérateur A est à domaine non dense et vérifie la condition de Hille-Yosida (0.2).

Notre contribution à l'étude des équations de type neutre à retard infini concerne la reproduction de quelques résultats concernant l'étude de l'équation (0.16) dans le cas où le retard est infini. Plus précisément, nous nous intéressons à l'équation suivante:

$$\begin{cases} \dfrac{d}{dt}\left[x(t) - G(t,x_t)\right] = A\left[x(t) - G(t,x_t)\right] + F(t,x_t), \quad t \ge 0, \\ x(\theta) = \varphi(\theta), \ \ -\infty < \theta \le 0, \end{cases} \tag{0.17}$$

avec $\varphi \in \mathcal{B}$, la fonction $G : [0, +\infty[\times \mathcal{B} \to E$ est continue et les mêmes notations que l'équation (0.1).
Signalons que Hernandez et Henriquez [61], [62] et [63] ont étudié l'équation (0.17) dans le cas où A est générateur d'un semigroupe analytique dans E et G est à valeurs dans le domaine d'une puissance fractionnaire de $-A$, $(D((-A)^{-\alpha}), 0 < \alpha < 1)$.

0.6 Description de la thèse

Dans la suite, nous donnons un résumé des résultats développés dans chaque chapitre.

Dans le chapitre 1, nous avons présenté quelques résultats d'analyse fonctionnelle qui seront utilisés le long de ce travail. La première section est une présentation axiomatique de la théorie fondamentale de l'espace de phase pour l'étude des équations différentielles à retard infini. Dans la deuxième section, nous avons donné des résultats sur la théorie des semi-groupes intégrés.

Dans le chapitre 2, nous avons étudié l'existence et la régularité des solutions pour l'équation (0.1), dans le cas où F est continue et Lipschitzienne par rapport à la deuxième variable.

Dans le chapitre 3, nous nous intéressons à l'aspect semi-groupe de la solution. Nous avons étudié l'existence et la régularité des solutions dans le cas où F est localement Lipschitzienne. Dans le cas où l'existence globale est vérifiée, nous avons démontré que la solution définit un semi-groupe qui vérifie la propriété de translation. Ceci nous a permis d'étudier la stabilité d'un point d'équilibre et de démontrer l'existence d'un attracteur global.

Dans le chapitre 4, nous avons donné des conditions suffisantes d'existence et de régularité pour l'équation (0.17). Dans la cas autonome, nous avons étudié la stabilité d'un point d'équilibre.

Dans le chapitre 5, nous avons traité l'existence de solutions périodiques pour l'équation (0.1) dans le cas où F est périodique en t. Dans le cas où F est linéaire par rapport à la deuxième variable, nous avons démontré que l'existence d'une solution bornée entraine l'existence d'une solution périodique.

Chapter 1

Phase Spaces and Integrated Semigroups

This chapter serves as an introduction to the other chapters. It contains the essential background materials required throughout this thesis. Its organization is as follows. In Section 1.1, we discuss the axiomatic phase space to functional differential equations with infinite delay. In Section 1.2, we collect some useful results on integrated semigroups' theory and differential operators with nondense domain. We only state results and leave the details to references.

We use $(E, |.|)$ or simply E, to denote a Banach space with norm $|.|$.

1.1 Phase space of differential equations with infinite delay

In the literature devoted to functional differential equations (FDEs) with finite delay ($r \geq 0$), the phase space, the space of initial data, is much of time the space of all continuous functions on $[-r, 0]$, endowed with the uniform norm topology. However, when the delay is infinite, the selection of the phase space plays an important role in the study of both qualitative and quantitative theory. Indeed, before 1966, every author chose a space that thought that it implies interesting properties of the equation under investigation. But, many repetitions have been pointed out. So, many trials have been done in order to avoid repetitions and summarize results. The purpose was the discussion of FDEs with infinite delay in an abstract phase space defined by some axioms. These axioms are only some properties of many concrete spaces used before. Effectively, the first axiomatic approach was given by B. D. Coleman and V. J. Mizel in [35]-[37]. After those papers, many contributions have been published by many authors until 1978 when J. K. Hale and J. Kato organized the study of functional differential equations with infinite delay in [55]. Which leads to many investigations on this theory. Our results are obtained as far as possible on the phase space introduced by those latter. That is, a seminormed space satisfying suitable axioms considered by Kappel and Schappacher [68], and Schumacher [111]. The book by Y. Hino and al. [66] contains an exhaustive bibliography and a detailed discussion on this subject. Following this book, we will assume that the phase space $\mathcal{B}$ is a linear space of functions mapping $(-\infty, 0]$ into $(E, |.|)$, endowed with a seminorm $\|.\|_{\mathcal{B}}$ and satisfying the following fundamental axioms:

(**A**) There exist a positive constant H and functions $K(\cdot), M(\cdot) : [0, +\infty) \to [0, +\infty)$, with K continuous and M locally bounded, such that for any $\sigma \in I\!R$ and $a > 0$, if $x : (-\infty, \sigma + a] \to E$, $x_\sigma \in \mathcal{B}$, and $x(\cdot)$ is continuous on $[\sigma, \sigma + a]$, then for all t in $[\sigma, \sigma + a]$ the following conditions hold:

- (**i**) $x_t \in \mathcal{B}$,
- (**ii**) $\|x(t)\| \leq H \|x_t\|_{\mathcal{B}}$,
- (**iii**) $\|x_t\|_{\mathcal{B}} \leq K(t - \sigma) \sup_{\sigma \leq s \leq t} \|x(s)\| + M(t - \sigma) \|x_\sigma\|_{\mathcal{B}}$.

(**A1**) For the function $x(\cdot)$ in (**A**), $t \mapsto x_t$ is a $\mathcal{B}$-valued continuous function for t in $[\sigma, \sigma + a]$.

(**B**) The space $\mathcal{B}$ is complete.

Remark 1.1.1 *[66]* ***(a)*** *Axiom* $(\mathbf{A} - \mathbf{ii})$ *is equivalent to*
$(\mathbf{A} - (\mathbf{ii})')$ $|\phi(0)| \leq H ||\phi||_{\mathcal{B}}$, *for every* $\phi \in \mathcal{B}$.
(b) *Since* $||.||_{\mathcal{B}}$ *is a seminorm, two elements* $\phi, \psi \in \mathcal{B}$ *can verify* $||\phi - \psi||_{\mathcal{B}} = 0$ *without necessarily* $\phi(\theta) = \psi(\theta)$ *for all* $\theta \leq 0$. *But, from* $\mathbf{A} - (\mathbf{ii})'$, *we see that*

$$\phi, \psi \in \mathcal{B} \quad \textit{and} \quad ||\phi - \psi||_{\mathcal{B}} = 0 \quad \textit{implies that} \quad \phi(0) = \psi(0).$$

Remark 1.1.2 *[66] Axiom* $(\mathbf{B})$ *is equivalent to saying that the space of equivalence classes* $\widehat{\mathcal{B}} := \mathcal{B} / \|.\|_{\mathcal{B}} = \left\{ \widehat{\phi} : \phi \in \mathcal{B} \right\}$ *is a Banach space.*

Let C_{00} be the set of continuous functions $\phi : (-\infty, 0] \to E$ with compact support *supp*(ϕ).

Theorem 1.1.1 *[66] Any $\phi \in C_{00}$ belongs to $\mathcal{B}$. If supp(ϕ) is contained in $[-r,-s]$, $0 \le s \le r < \infty$, then there exists a constant $\delta(r,s)$ such that*

$$\|\phi\|_{\mathcal{B}} \le \delta(r,s) \sup_{\theta\in[-r,-s]} |\phi(\theta)|.$$

Let us give some examples of concrete functional spaces that verifies Axioms (**A**), (**A1**) and (**B**).

Example 1.1.1 *For any continuous function $g : (-\infty, 0] \to (0, +\infty)$, let*

$$C_g^0 := \{\phi \in C((-\infty,0]; E) : \lim_{\theta\to-\infty} \frac{|\phi(\theta)|}{g(\theta)} = 0\},$$

endowed with the norm

$$\|\phi\|_g := \sup_{-\infty<\theta\le 0} \frac{|\phi(\theta)|}{g(\theta)}.$$

It was proved in Theorems 1.**3.2** and 1.**3.6** of [66] that if g is nonincreasing, then $\left(C_g^0, \|\cdot\|_g\right)$ satisfies Axioms (**A**), (**A1**) and (**B**).

Example 1.1.2 *Consider the space $C_r \times L^1(g)$ of all functions $\phi : (-\infty,0] \to E$ such that ϕ is continuous on $[-r,0]$, for some $r \ge 0$, Lebesgue measurable and $g(.)\,|\phi(.)|$ is Lebesgue integrable on $(-\infty,-r)$, where $g : (-\infty,-r) \to I\!R$ is a positive Lebesgue measurable function. A seminorm in $C_r \times L^1(g)$ is defined by*

$$\|\phi\|_r := \sup_{-r\le\theta\le 0} |\phi(\theta)| + \int_{-\infty}^{-r} g(\theta)\,|\phi(\theta)|\,d\theta.$$

We suppose that g satisfies :
(i) g is integrable on $(-d,-r)$ for any $d \ge r$, and
(ii) there exists a locally bounded function $G : (-\infty,0] \to [0,+\infty)$ such that

$$g(\xi+\theta) \le G(\xi)g(\theta), \text{ for all } \xi \le 0 \text{ and } \theta \in (-\infty,-r)\setminus N_\xi,$$

where $N_\xi \subseteq (-\infty,-r)$ is a set with Lebesgue measure 0.
*Thus, Theorem **1**.3.8 in [66] asserts that $C_r \times L^1(g)$ is a phase space which verifies Axioms (**A**), (**A1**) and (**B**).*

To study the existence of periodic solutions in Chapter 5, we need to introduce an additional axiom which is concerned with the realization of some elements in $\mathcal{B}$. Recall that a sequence of functions $(\phi^n)_{n\in \boldsymbol{N}} : (-\infty, 0] \to E$, is said uniformly bounded if $\sup_{n\in \boldsymbol{N}} \left(\sup_{-\infty<\theta\leq 0} |\phi^n(\theta)| \right) < +\infty$. We say that a sequence of functions $(\phi_n)_{n\geq 0} \in \mathcal{B}$ converges compactly on $(-\infty, 0]$ to ϕ if the sequence converges uniformly on compact subsets of $(-\infty, 0]$.

(**C**) If a uniformly bounded sequence $(\phi^n)_n$ in C_{00} converges to a function ϕ compactly on $(-\infty, 0]$, then ϕ is in $\mathcal{B}$ and $\|\phi^n - \phi\|_{\mathcal{B}} \to 0$ as $n \to +\infty$.

Section 1.3 of the book [66] contains an exhaustive discussion on the standard spaces that satisfy most axioms used in this thesis. For example, it is known in Theorem 1.**3.2** of this book that the space C_g^0 introduced in Example 1.1.1 satisfies Axiom (**C**) if g is nonincreasing and $g(s) \to \infty$ as $s \to -\infty$.

More recently, Ruess and Summers in [107] have considered another axiom that we use in this thesis to deal with existence of strict solutions and characterization of the generator of the linear solution semigroup.

(**D1**) For a sequence $(\phi_n)_{n\geq 0}$ in $\mathcal{B}$, if $\|\phi_n\|_{\mathcal{B}} \to 0$ then $|\phi_n(s)| \to 0$ for each $s \in (-\infty, 0]$.

Example 1.1.3 *As an example, we can verify without difficulties that the above axiom is satisfied by the space C_g^0 in the special case $g(\theta) = e^{-\gamma\theta}$, $\gamma > 0$:*

$$C_\gamma^0 = \left\{ \phi \in C((-\infty, 0]; E) : \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) = 0 \right\}, \ \gamma > 0,$$

with the norm $\|\phi\|_\gamma = \sup_{\theta\leq 0} e^{\gamma\theta} |\phi(\theta)|$, $\phi \in C_\gamma^0$. It is satisfied, in general, by the space

$$C_\gamma = \left\{ \phi \in C((-\infty, 0]; E) : \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) \text{ exists in } E \right\}, \ \gamma > 0.$$

For $\phi \in \mathcal{B}$, $t \geq 0$ and $\theta \leq 0$, we define

$$[W(t)\phi](\theta) = \begin{cases} \phi(0) \text{ if } t + \theta \geq 0 \\ \phi(t+\theta) \text{ if } t + \theta < 0. \end{cases}$$

We can see that $(W(t))_{t\geq 0}$ is a C_0-semigroup on $\mathcal{B}$. We set

$$W_0(t) = W(t)_{/\mathcal{B}_0}, \text{ where } \mathcal{B}_0 := \{\phi \in \mathcal{B} : \phi(0) = 0\}.$$

$\mathcal{B}$ is called a fading memory space if it satisfies the extra axiom (**C**) and

($\mathbf{D2}$) $\|W_0(t)\phi\|_{\mathcal{B}} \to 0$ as $t \to +\infty$ for all $\phi \in \mathcal{B}_0$.

Let BC be the space of bounded continuous functions mapping $(-\infty, 0]$ into E, provided with the uniform norm topology, then one has.

Proposition 1.1.2 *[66] Assuming that $\mathcal{B}$ is a fading memory space, then $BC \subset \mathcal{B}$ and there exists a positive constant J such that $\|\phi\|_{\mathcal{B}} \leq J \|\phi\|_{BC}$. Moreover*

$$\|x_t\|_{\mathcal{B}} \leq J \sup_{\sigma \leq s \leq t} |x(s)| + (1 + JH) \|W_0(t - \sigma)\| \|x_\sigma\|_{\mathcal{B}}, \ \sigma > 0,$$

for any function x arising in Axiom ($\mathbf{A}$).

As a consequence of this proposition, the functions $K(.)$ and $M(.)$ can be chosen as $K(t) = J$ and $M(t) = (1 + JH) \|W_0(t)\|$. Note that ($\mathbf{D2}$) implies $\sup_{t \geq 0} \|W_0(t)\phi\|_{\mathcal{B}} < +\infty$ by the Banach-Steinhaus theorem. Therefore , whenever $\mathcal{B}$ is a fading memory space, the functions $K(.)$ and $M(.)$ can be assumed bounded on $[0, +\infty)$.

$\mathcal{B}$ is called a uniform fading memory space if it satisfies the extra axioms ($\mathbf{C}$) and ($\mathbf{D3}$) $\|W_0(t)\| \to 0$ as $t \to +\infty$.

One can see that Axiom ($\mathbf{D3}$) holds for $\mathcal{B} = C_g^0$ if and only if $\sup \left\{ \dfrac{g(s+t)}{g(s)} : s \leq -t \right\} \to 0$ as $t \to \infty$. Therefore, C_γ^0 is a uniform fading memory space. On the other hand, if $g(s) = 1 + |s|^k$ for some $k > 0$, then the space C_g^0 is a fading memory space, but not a uniform fading memory space.

We will use Axioms ($\mathbf{C}$), ($\mathbf{D2}$) and ($\mathbf{D3}$) in studying existence of periodic solutions (Chapter 5). One can see Chapter 7 of [66] for an earlier use in discussing stability and existence of periodic solutions or almost periodic solutions to FDEs with infinite delay.

($\mathbf{C1}$) If $(\phi_n)_{n \geq 0}$ is a Cauchy sequence in $\mathcal{B}$ with respect to the seminorm and if $(\phi_n)_{n \geq 0}$ converges compactly to ϕ on $(-\infty, 0]$, then ϕ is in $\mathcal{B}$ and $\|\phi_n - \phi\|_{\mathcal{B}} \to 0$.

The above axiom is also concerned by the realization of elements in $\mathcal{B}$. It is sometimes considered in stability investigation additionally to the following axiom :

($\mathbf{C2}$) There exists a constant γ_0 such that the function $(\varepsilon_\lambda \quad u) : (-\infty, 0] \to E$ defined by $e^{\lambda \theta} u$ for $\theta \leq 0$, belongs to $\mathcal{B}$ for $Re(\lambda) > \gamma_0$ and $u \in E$, and that

$$\|\varepsilon_\lambda\| := \sup \{ \|\varepsilon_\lambda \quad u\| : u \in E, \ \ |u| \leq 1 \}$$

is finite for each λ with $Re(\lambda) > \gamma_0$, and bounded for $Re(\lambda) > \gamma_1$ for some $\gamma_1 \geq \gamma_0$.

One of the important consequence of Axiom ($\mathbf{C1}$) is that it allows computing the integral in $\mathcal{B}$ from the integral in E (see Lemma 2.2.6 in Chapter 2 or [91]). Here, we use Axiom

(**C1**) in studying existence of strict solutions.
Some applications of Axioms (**C**), (**C1**) and (**C2**) can be found in [56] and [66] and most recently [91], [115], [92] and [114] where they were essentially used in studying stability and periodicity of solutions to FDEs with infinite delay.

1.2 Integrated semigroups and differential operators with nondense domain

In this section, we recall some materials about differential operators with nondense domain, integrated semigroups' theory and its applications to abstract Cauchy problems. We will only state results without proofs. See the references [18], [19], [32], [40], [64], [71] and [116] for more details, and the articles by Adimy and Ezzinbi for a summary of this theory.

It is well known that the semigroup method permits treating a large class of Cauchy problems such as :

$$\begin{cases} \dfrac{d}{dt}u(t) = Au(t) + f(t), \ t \geq 0, \\ u(0) = x \in E, \end{cases} \tag{1.1}$$

where A is an unbounded linear operator of domain $D(A) \subseteq E$ (f and x are given). In order that this method can be useful, it is necessary that A be the infinitesimal generator of a C_0-semigroup, that is, it satisfies the two conditions of application of the Hille-Yosida theorem :

(**i**) $\overline{D(A)} = E$,

(**ii**) there exist $\bar{M} \geq 0$, $\omega \in \mathbb{R}$ such that if $\lambda > \omega$ then $(\lambda I - A)^{-1} \in \mathcal{L}(E)$ and

$$\left\|(\lambda - \omega)^n (\lambda I - A)^{-n}\right\| \leq \bar{M}, \quad \text{for all } n \in I\!N,$$

where $\mathcal{L}(E)$ is the space of bounded linear operators from E into E.
However, nondensity of $D(A)$ into E occurs in many situations due to restrictions on the space where the problems are considered (for example, periodic continuous functions, Hölder continuous functions) or due to boundary conditions (for example, the space C^1 with null value on the boundary is non dense in the space of continuous functions) (see the examples given in [40] at the end of this chapter).

It's true that when the function f is equal to zero, the Cauchy problem (1.1) can still be handled by using the classical semigroup theory because A generates a strongly continuous semigroup in the space $\overline{D(A)}$. But, when $f \neq 0$ it is necessary to impose additional restrictions, the simplest of which is that f takes its values in $\overline{D(A)}$. It is the integrated semigroups theory that allows the range of the operator f to be a subset of E not necessarily contained in $\overline{D(A)}$.

(Once) integrated semigroups are motivated by formally defining

$$S(t) = \int_0^t T(s)ds,$$

with a C_0-semigroup $(T(t))_{t\geq 0}$ and discovering that

$$S(0) = 0 \text{ and } S(s)S(t) = \int_0^{t+s} S(\tau)d\tau - \int_0^{s} S(\tau))d\tau - \int_0^{t} S(\tau))d\tau, \quad t, s \geq 0.$$

The following definitions are due to Arendt.

Definition 1.2.1 *[18] A family $(S(t))_{t\geq 0} \subset \mathcal{L}(E)$ is called an integrated semigroup if the following conditions are satisfied :*

(i) $S(0) = 0$*;*

(ii) *for any $x \in E$, $S(t)x$ is a continuous function of $t \geq 0$ with values in E;*

(iii) *for any $t, s \geq 0$ $\quad S(s)S(t) = \int_0^s (S(t+\tau) - S(\tau))d\tau.$*

Definition 1.2.2 *[18] An integrated semigroup $(S(t))_{t\geq 0}$ is called exponentially bounded, if there exist constants $\bar{M} \geq 0$ and $\omega \in I\!R$ such that*

$$\|S(t)\| \leq \bar{M}e^{\omega t}, \text{ for } t \geq 0.$$

Moreover, $(S(t))_{t\geq 0}$ is called nondegenerate if $S(t)x = 0$, for all $t \geq 0$, implies that $x = 0$.

If $(S(t))_{t\geq 0}$ is an integrated semigroup, exponentially bounded, then the Laplace transform $\mathcal{R}(\lambda) := \lambda \int_0^{+\infty} e^{-\lambda t} S(t)dt$ exists for all λ with $Re(\lambda) > \omega$. $\mathcal{R}(\lambda)$ is injective if and only if $(S(t))_{t\geq 0}$ is nondegenerate. $\mathcal{R}(\lambda)$ satisfies the following expression

$$\mathcal{R}(\lambda) - \mathcal{R}(\mu) = (\mu - \lambda)\mathcal{R}(\lambda)\mathcal{R}(\mu).$$

It's well known from [97] that if $\mathcal{R}(\lambda)$ is injective, then there exists a unique operator A satisfying $(\omega, +\infty) \subset \rho(A)$ (the resolvent set of A) such that

$$\mathcal{R}(\lambda) = R(\lambda, A) := (\lambda I - A)^{-1}, \text{ for all } Re(\lambda) > \omega.$$

This operator A is called the generator of $(S(t))_{t\geq 0}$.
We have the following general definition.

Definition 1.2.3 *[18] A linear operator $A : D(A) \subset E \to E$ is called a generator of an integrated semigroup, if there exists $\omega \in I\!R$ such that $(\omega, +\infty) \subset \rho(A)$, and there exists a strongly continuous exponentially bounded family $(S(t))_{t\geq 0}$ of linear bounded operators such that $S(0) = 0$ and $(\lambda I - A)^{-1} = \lambda \int_0^{+\infty} e^{-\lambda t} S(t)dt$ for all $\lambda > \omega$.*

Similar results as for semigroups can be obtained for integrated semigroups.

Proposition 1.2.1 *[18] Let A be the generator of an integrated semigroup $(S(t))_{t\geq 0}$. Then for all $x \in E$ and $t \geq 0$,*

$$\int_0^t S(s)xds \in D(A) \quad \text{and} \quad S(t)x = A\left(\int_0^t S(s)xds\right) + tx.$$

Moreover, for all $x \in D(A)$, $t \geq 0$

$$S(t)x \in D(A), \qquad\qquad AS(t)x = S(t)Ax$$

and

$$S(t)x = \int_0^t S(s)Ax\,ds + tx.$$

Corollary 1.2.2 *[18] Let A be the generator of an integrated semigroup $(S(t))_{t\geq 0}$. Then for all $x \in E$ and $t \geq 0$ one has $S(t)x \in \overline{D(A)}$.*
Moreover, let $x \in E$. Then $S(\cdot)x$ is right-sided differentiable in $t \geq 0$ if and only if $S(t)x \in D(A)$. In that case

$$S'(t)x = AS(t)x + x.$$

One other result is useful.

Proposition 1.2.3 *[64] Let $A : D(A) \subset E \to E$ be a linear operator and $(S(t))_{t\geq 0} \subset \mathcal{L}(E)$ an exponentially bounded family. The following assertions are equivalent*
(i) $\int_0^t S(s)xds \in D(A)$ *and* $S(t)x = A\left(\int_0^t S(s)xds\right) + tx, \quad (t \geq 0, x \in E),$
(ii) *$(S(t))_{t\geq 0}$ is an integrated semigroup on E generated by A.*

An important special case is when the integrated semigroup is locally Lipschitz continuous (with respect to time).

Definition 1.2.4 *[71] An integrated semigroup* $(S(t))_{t\geq 0}$ *is called locally Lipschitz continuous, if for all* $a>0$ *there exists a constant* $l(a)>0$ *such that*

$$\|S(t)-S(s)\| \leq l(a)\,|t-s|\,,\ \text{for all } t,s\in[0,a]\,.$$

In this case, we know from [71], that $(S(t))_{t\geq 0}$ is exponentially bounded.

Definition 1.2.5 *[71] We say that a linear operator* $A : D(A)\subset E\to E$ *satisfies the Hille-Yosida condition* (HY) *if there exist* $\bar{M}\geq 0$ *and* $\omega\in \mathbb{R}$ *such that* $(\omega,+\infty)\subset\rho(A)$ *and*

$$\sup\left\{(\lambda-\omega)^n\left\|R(\lambda,A)^n\right\|,\ n\in\mathbb{N},\ \lambda>\omega\right\}\leq\bar{M}. \qquad (HY)$$

The following theorem shows that the Hille-Yosida condition (HY) characterizes generators of locally Lipschitz continuous integrated semigroups.

Theorem 1.2.4 *[71] The following assertions are equivalent.*
(i) *A is the generator of a locally Lipschitz continuous non degenerate integrated semigroup,*
(ii) *A satisfies the condition* (HY)*.*

In addition, $l(\tau)$ in Definition 1.2.4 can be taken such that $l(\tau)\leq\bar{M}^2e^{\omega\tau}$.

Proposition 1.2.5 *[71] Let* $A : D(A)\subset E\to E$ *be the generator of a locally Lipschitz continuous integrated semigroup* $(S(t))_{t\geq 0}$*. If we set for all* $\lambda\in\mathbb{R}$,

$$\begin{cases} B: D(A)\subset E\to E,\\ Bu=Au-\lambda u,\end{cases}$$

then B is the generator of the locally Lipschitz continuous integrated semigroup $(S_\lambda(t))_{t\geq 0}$ *given by*

$$S_\lambda(t)=e^{-\lambda t}S(t)+\lambda\int_0^t e^{-\lambda s}S(s)\,ds.$$

Moreover, in relation with the above theorem and Definition 1.2.5, B satisfies (HY) *with* $\omega_B=\omega_A-\lambda$.

Theorem 1.2.6 *Let A be a Hille-Yosida operator and* $(S(t))_{t\geq 0}$ *be the locally Lipschitz continuous integrated semigroup generated by A. It is well known from [117], that the derivative* $(S'(t))_{t\geq 0}$ *on* $\overline{D(A)}$ *is a strongly continuous semigroup generated by the part* A_0 *of the operator A in* $\overline{D(A)}$*, which is defined by*

$$\begin{cases} D(A_0)=\left\{x\in D(A) : Ax\in\overline{D(A)}\right\},\\ A_0x=Ax,\ \text{for } x\in D(A_0).\end{cases}$$

Throughout this text, a linear operator that satisfies the Hille-Yosida condition (HY) without being necessarily densely defined is called a Hille-Yosida operator.

Next, we give some results for the existence of solutions to the following Cauchy problem

$$\begin{cases} \dfrac{d}{dt}u(t) = Au(t) + f(t), \quad t \geq 0, \\ u(0) = x \in E, \end{cases} \tag{1.2}$$

where A is a closed linear operator on E.

The following results are due to Da Prato and Sinestrari.

Definition 1.2.6 *[40] We say that a function $u : [0, a] \to E$, $a > 0$, is a strict solution of Equation (1.2) in $[0, a]$ if the following conditions hold :*
(i) $u \in C^1([0, a]\,; E) \cap C([0, a]\,; D(A))$;
(ii) *u satisfies Equation (1.2) on $[0, a]$.*

Theorem 1.2.7 *[40] Let $A : D(A) \subseteq E \to E$ be a linear operator, $f : [0, a] \to E$, $x \in D(A)$ such that*
(i) *A is a Hille-Yosida operator,*
(ii) *$f(t) = f(0) + \int_0^t g(s)\, ds$ for some Bochner-integrable function g,*
(iii) $Ax + f(0) \in \overline{D(A)}$.
Then there exists a unique strict solution u of Equation (1.2) on the interval $[0, a]$, and for each $t \in [0, a]$

$$|u(t)| \leq \bar{M} e^{\omega t} \left(|x| + \int_0^t e^{-\omega s} |f(s)|\, ds \right). \tag{1.3}$$

In the case where x is not sufficiently regular (that is, x is just in $\overline{D(A)}$) there may not exist a strict solution $u(t) \in E$ but, following the work of Da Prato and Sinestrari [40], Equation (1.2) may still have a so-called integral solution. This motivates the following definition:

Definition 1.2.7 *[40] Given $f \in L^1_{loc}(0, +\infty; E)$ and $x \in E$, we say that $u : [0, +\infty) \to E$ is an integral solution of Equation (1.2) if the following assertions are true*
(i) $u \in C([0, +\infty)\,; E)$,
(ii) $\int_0^t u(s)ds \in D(A)$, *for* $t \geq 0$,
(iii) $u(t) = x + A\left(\int_0^t u(s)ds\right) + \int_0^t f(s)ds$, *for* $t \geq 0$.

From this definition, we deduce that for an integral solution u, we have $u(t) \in \overline{D(A)}$, for all $t > 0$, because $u(t) = \lim_{h\to 0} \frac{1}{h} \int_t^{t+h} u(s)ds$ and $\int_t^{t+h} u(s)ds \in D(A)$. In particular, $x \in \overline{D(A)}$ is a necessary condition for the existence of an integral solution to Equation (1.2).

Theorem 1.2.8 *([32], [40]) Suppose that A is a Hille-Yosida operator, $x \in \overline{D(A)}$ and $f : [0, +\infty) \to E$ is a continuous function. Then the problem (1.2) has a unique integral solution which is given by the following variation of constants formula*

$$u(t) = S'(t)x + \frac{d}{dt}\left(\int_0^t S(t-s)f(s)ds\right), \quad \text{for } t \geq 0, \tag{1.4}$$

where $S(t)$ is the integrated semigroup generated by A.
Furthermore, the function u satisfies the inequality (1.3).

The above theorem says also that $\int_0^t S(t-s)f(s)ds$ is differentiable with respect to t.
The following result is needed throughout this thesis.

Proposition 1.2.9 *([1], [71] and [116]) Let $A : D(A) \subseteq E \to E$ be a Hille-Yosida operator, $(S(t))_{t\geq 0}$ be the integrated semigroup generated by A and $G : [0, a] \to E$, $a > 0$, be a Bochner-integrable function. Then, the function $B : [0, a] \to E$ defined by*

$$B(t) = \int_0^t S(t-s)G(s)\, ds$$

is continuously differentiable on $[0, a]$ and satisfies for $t \in [0, a]$,
(i) $B'(t) = \lim_{h\searrow 0} \frac{1}{h} \int_0^t S'(t-s)S(h)G(s)\, ds,$
(ii) $B(t) \in D(A),$
(iii) $B'(t) = AB(t) + \int_0^t G(s)\, ds.$
(iv) $\mathcal{R}(\lambda)B'(t) = \int_0^t S'(t-s)\mathcal{R}(\lambda)G(s)\, ds$, *for* $\lambda > \omega$,
(v) $|B'(t)| \leq 2l \int_0^t |G(s)|\, ds,$
(vi) $B'(t) = \lim_{\lambda\to+\infty} \int_0^t S'(t-s)B_\lambda G(s)\, ds,$
where $B_\lambda := \lambda\mathcal{R}(\lambda)$ and $l = l(a)$ is the Lipschitz constant of $S(.)$ on $[0, a]$.

Let us end with some classical examples of Hille-Yosida operators with nondense domain (for more details, see [40]).

1)

$$\begin{cases} E = C([0,a]\,, I\!R), \\ D(A) = \left\{u \in C^1([0,a]\,, I\!R);\ u(0) = 0\right\}, \\ Au = -u'. \end{cases}$$

2)

$$\begin{cases} E = \{u \in C^\alpha([0,a]\,, I\!R);\ u(0) = 0\}\,, \quad 0 < \alpha < 1, \\ D(A) = \left\{u \in C^{1+\alpha}([0,a]\,, I\!R);\ u(0) = u'(0) = 0\right\}, \\ Au = -u', \end{cases}$$

where

$$C^\alpha([0,a]\,, I\!R) = \left\{u : [0,a] \to I\!R;\ [u]_{C^\alpha([0,a],\boldsymbol{R})} = \sup_{0 \le t \le s \le a} \frac{|u(t) - u(s)|}{|t-s|^\alpha} < \infty\right\},$$

and

$$C^{1+\alpha}([0,a]\,, I\!R) = \left\{u : [0,a] \to I\!R;\ u' \in C^\alpha([0,a]\,, I\!R)\right\},$$

with

$$\|u\|_{C^\alpha([0,a],\boldsymbol{R})} = \|u\|_{C([0,a],\boldsymbol{R})} + [u]_{C^\alpha([0,a],\boldsymbol{R})}\,.$$

3)

$$\begin{cases} E = C([0,a]\,, I\!R), \\ D(A) = \left\{u \in C^2([0,a]\,, I\!R);\ u(0) = u(a) = 0\right\}, \\ Au = u''. \end{cases}$$

4) Let   be a bounded open set of $I\!R^n$ with regular boundary Γ. Define

$$\begin{cases} E = C(\overline{\ }\,, I\!R), \\ D(A) = \left\{u \in C(\overline{\ }\,, I\!R);\ u = 0 \text{ on } \Gamma \text{ and } \Delta u \in C(\overline{\ }\,, I\!R)\right\}, \\ Au = \Delta u, \end{cases}$$

here Δ is the Laplacian in the sense of distributions on   .

5) Let

$$
\begin{cases}
E = W_0^{1,p}(0,a; C(\overline{\ }, I\!\!R)), \ 1 \le p < \infty, \\
D(A) = \left\{ \begin{array}{c} u \in C([0,a]\,, D(\Delta)) \cap C^1([0,a]\,, C^1(\overline{\ }, I\!\!R)); \\ u(0) = 0 \text{ and } \Delta u - u' \in W_0^{1,p}(0,a; C(\overline{\ }, I\!\!R)) \end{array} \right\}, \\
Au = \Delta u - u'.
\end{cases}
$$

where

$$
W_0^{1,p}(0,a;E) = \left\{ u : [0,a] \to E; \ u(t) = \int_0^t u'(s)ds \text{ and } u' \in L^p(0,a;E) \right\}.
$$

6) Let

$$
\begin{cases}
E = W_0^{1,p}(0,a; C(\overline{\ }, I\!\!R)), \\
D(A) = \left\{ \begin{array}{c} u \in C([0,a]\,, D(\Delta)) \cap C^1([0,a]\,, C(\overline{\ }, I\!\!R)); \\ u(0) = 0 \text{ and } \Delta u - u' \in W_0^{1,p}(0,a; C(\overline{\ }, I\!\!R)) \end{array} \right\}, \\
Au = \Delta u - u'.
\end{cases}
$$

One can also consider the periodic versions of examples **5** and **6**.

7)

$$
\begin{cases}
E = L^\infty(I\!\!R), \\
D(A) = \left\{ u \in L^\infty(I\!\!R), \ u \text{ is absolutely continuous and } u' \in L^\infty(I\!\!R) \right\}, \\
Au = -u'.
\end{cases}
$$

Chapter 2

Global Existence and Regularity of Solutions for Some Partial Functional Differential Equations with Infinite Delay[1]

2.1 Introduction

Let $(E, |.|)$ be a Banach space. In all this thesis, $\mathcal{B}$ is a linear space of functions mapping $(-\infty, 0]$ into E, endowed with a seminorm $\|.\|_{\mathcal{B}}$ and satisfying the fundamental axioms (**A**), (**A1**) and (**B**) introduced in the previous chapter. Since, in Axiom (**A**) the functions $K(.)$ and $M(.)$ are, respectively, continuous and locally bounded, we will use the notations K_a and M_a to denote, respectively, $\max\limits_{0\leq t\leq a} K(t)$ and $\sup\limits_{0\leq t\leq a} M(t)$ for a fixed constant $a > 0$.

The present chapter is devoted to some existence results to the following class of partial functional differential equations (PFDEs) with infinite delay

$$\begin{cases} \dfrac{d}{dt}x(t) = Ax(t) + F(t, x_t), \quad t \geq 0, \\ x_0 = \varphi, \end{cases} \tag{2.1}$$

where $A : D(A) \subset E \to E$ is a closed linear operator, φ is an element of $\mathcal{B}$, F is an appropriate mapping defined on $[0, +\infty) \times \mathcal{B}$ with values in E and, for each $x : (-\infty, b] \to E$,

[1] This chapter is based on a paper in collaboration with M. Adimy and K. Ezzinbi. The paper will appear in Journal of Nonlinear Analysis, Theory, Methods and Applications, (2001).

$b > 0$, and $t \in [0, b]$, x_t represents the mapping defined from $(-\infty, 0]$ into E by

$$x_t(\theta) = x(t+\theta), \quad \text{for } \theta \in (-\infty, 0].$$

Surely there are works dealing with existence of solutions to Equation (2.1) with dense domain $D(A)$ (cf., for instance, [56], [66], [124], [91] and [58]-[60]). However, most existing results concern the case where $A = 0$ and the extension to the case of non-dense domain looks quite natural. We state that we can solve Equation (2.1) without assuming necessarily that A is densely defined. Some of our results are an extension of Henriquez's ones ([58]-[60]) to the situation where A satisfies the resolvent estimates of the well-known Hille and Yosida theorem while the domain of A is not dense in E. Techniques employed in [1], [7] and [45], have been generalized to the study of Equation (2.1). That is, under different appropriate assumptions, existence and uniqueness of solutions (called integral solutions) are first proved, using a fixed point argument. Then, the integral solutions are shown to be strict solutions under more restrictive assumptions.

The main tool in the approach followed in our proves is the theory of integrated semigroups introduced in Section 1.2 in Chapter 1.

2.2 Existence and regularity of solutions

Throughout this text, we assume that
(**H2.1**) $A : D(A) \subset E \to E$ is a Hille-Yosida operator with constants $\bar{M} \geq 1$ and $\omega \in I\!R$.

Suppose that F is continuous on $[0, +\infty) \times \mathcal{B}$ with values in E. The initial value problem associated with Equation (2.1) is the following: for given $\varphi \in \mathcal{B}$, find a continuous function $x : (-\infty, a] \to E$, $a > 0$, differentiable on $[0, a]$ such that $x(t) \in D(A)$, for $t \in [0, a]$ and x satisfies Equation (2.1), i.e.,

$$\begin{cases} \dfrac{d}{dt}x(t) = Ax(t) + F(t, x_t), & t \in [0, a], \\ x(t) = \varphi(t), & -\infty < t \leq 0. \end{cases}$$

In [60], it has been shown the following result:

Theorem 2.2.1 *[60] Assuming that A is the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on E, and $F : [0, a] \times \mathcal{B} \to E$ is a continuous mapping in t and uniformly Lipschitz continuous on $\mathcal{B}$. Then, for given $\varphi \in \mathcal{B}$, the Cauchy problem (2.1) has exactly one mild* *solution which is given by*

$$x(t) = \begin{cases} T(t)\varphi(0) + \int_0^t T(t-s)F(s,x_s)ds, & 0 \le t \le a, \\ \varphi(t), \quad -\infty < t \le 0. \end{cases}$$

We start by introducing the following definitions.

Definition 2.2.1 *Let $\varphi \in \mathcal{B}$. We say that a function $x : (-\infty, a] \to E$, $a > 0$, is an integral solution of Equation (2.1) in $(-\infty, a]$ if the following conditions hold :*
(i) x *is continuous on* $[0, a]$*;*
(ii) $\int_0^t x(s)ds \in D(A)$, *for* $t \in [0, a]$;
(iii) $x(t) = \begin{cases} \varphi(0) + A\int_0^t x(s)ds + \int_0^t F(s,x_s)ds, & 0 \le t \le a, \\ \varphi(t), \quad -\infty < t \le 0. \end{cases}$

Definition 2.2.2 *Let $\varphi \in \mathcal{B}$. We say that a function $x : (-\infty, a] \to E$ is a strict solution of Equation (2.1) in $(-\infty, a]$ if the following conditions hold :*
(i) $x \in C^1([0,a]\,; E) \cap C([0,a]\,; D(A))$;
(ii) x *satisfies Equation (2.1) on* $(-\infty, a]$.

From the closedness property of the operator A, we can prove the following.

Proposition 2.2.2 **(i)** *If x is an integral solution of Equation (2.1) in $(-\infty, a]$, then for all $t \in [0, a]$, $x(t) \in \overline{D(A)}$. In particular $\varphi(0) \in \overline{D(A)}$.*
(ii) *If x is an integral solution of Equation (2.1) in $(-\infty, a]$, such that $x : [0, a] \to E$ belongs to $C^1([0,a]\,; E)$ or $C([0,a]\,; D(A))$, then x is also a strict solution of Equation (2.1) in $(-\infty, a]$.*

Proof. **(i)** is an immediate consequence of the definition 2.2.1 of an integral solution x. In fact, since $x(t) = \lim_{h\to 0} \frac{1}{h}\int_t^{t+h} x(s)ds$ and $\int_t^{t+h} x(s)ds \in D(A)$ for all $t > 0$, we have for all $t \in [0, a]$, $x(t) \in \overline{D(A)}$. To prove **(ii)**, suppose that x is an integral solution of Equation (2.1) in $(-\infty, a]$, such that $x : [0, a] \to E$ belongs to $C^1([0,a]\,; E)$. Then, from Definition 2.2.1, for all $t \in [0, a]$ and $h > 0$,

$$A\frac{1}{h}\int_t^{t+h} x(s)ds = \frac{1}{h}(x(t+h) - x(t)) - \frac{1}{h}\int_t^{t+h} F(s,x_s)ds.$$

Since F is continuous, then the right side of the above equality tends to

$$\frac{d}{dt}x(t) - F(t, x_t), \text{ as } h \text{ tends to } 0^+, \text{ and}$$

$$\frac{1}{h}\int_t^{t+h} x(s)ds \text{ tends to } x(t), \text{ as } h \text{ tends to } 0^+.$$

From the closedness of A, we get $x(t) \in D(A)$ and

$$Ax(t) = \frac{d}{dt}x(t) - F(t, x_t).$$

From what we deduce that x is a strict solution.

On the other hand, if x belongs to $C([0, a]; D(A))$. Again from Definition 2.2.1, for all $t \in [0, a]$ and $h > 0$,

$$\frac{1}{h}(x(t+h) - x(t)) = \frac{1}{h}\int_t^{t+h} Ax(s)ds + \frac{1}{h}\int_t^{t+h} F(s, x_s)ds.$$

Since $Ax(.)$ and F are continuous, the right side of the above equality tends to

$$Ax(t) + F(t, x_t), \text{ as } h \text{ tends to } 0^+.$$

Which implies that x is differentiable at the right in t and

$$\frac{d^+}{dt}x(t) = Ax(t) + F(t, x_t).$$

Since $Ax(t) + F(t, x_t)$ is continuous on $[0, a]$, we conclude that x is differentiable on $[0, a]$ and $\frac{d}{dt}x(t) = Ax(t) + F(t, x_t)$. This finishes the proof of the Proposition. ■

2.2.1 Local existence and global continuation of integral solutions

Let be a nonempty open subset of $\mathcal{B}$.

Observe that Theorem 1.2.4 implies that, under assumption (**H2.1**), A is the generator of a locally Lipschitz continuous integrated semigroup $(S(t))_{t\geq 0}$ on E. In addition, $S'(t) : \overline{D(A)} \to \overline{D(A)}$ is a C_0-semigroup satisfying

$$\left|S'(t)y\right| \leq \bar{M}e^{\omega t}\left|y\right|, \quad \text{for all } t \geq 0 \text{ and } y \in \overline{D(A)}.$$

To obtain our first two results, we make the following compactness property of $(S'(t))_{t\geq 0}$.

(**H2.2**) The semigroup $(S'(t))_{t\geq 0}$ is compact on $\left(\overline{D(A)}, |\cdot|\right)$. It means that the operator $S'(t)$ is compact on $\overline{D(A)}$ whenever $t > 0$, .

Theorem 2.2.3 *Let* $F : [0,a] \times \quad \to E$ *be a continuous mapping and suppose that the conditions* (**H2.1**) *and* (**H2.2**) *hold. If* $\varphi \in \quad$ *with* $\varphi(0) \in \overline{D(A)}$*, then Equation (2.1) has at least one integral solution* $x : (-\infty, b] \to E$*, for some* $b \in (0,a]$ *. Moreover,*

$$x(t) = \begin{cases} S'(t)\varphi(0) + \dfrac{d}{dt}\displaystyle\int_0^t S(t-s)F(s,x_s)ds, & 0 \le t \le b, \\ \varphi(t), \quad -\infty < t \le 0. \end{cases}$$

Proof. The main idea of the proof is to use the Schauder fixed point theorem.
Let $\varphi \in \quad$ with $\varphi(0) \in \overline{D(A)}$. First, there exist constants $r_1 > 0$ and $\mu \ge 0$ such that $\overline{B_{r_1}(\varphi)} = \{\psi \in \mathcal{B} : \|\psi - \varphi\|_{\mathcal{B}} \le r_1\} \subseteq \quad$ and $|F(s,\psi)| \le \mu$ for all $s \in [0, r_1]$ and $\psi \in \overline{B_{r_1}(\varphi)}$. Consider the function $y : (-\infty, +\infty) \to E$ defined by

$$y(t) = \begin{cases} S'(t)\varphi(0), & \text{for } t \ge 0, \\ \varphi(t), & \text{for } t \le 0. \end{cases}$$

By virtue of Axioms $(\mathbf{A} - \mathbf{i})$ and $(\mathbf{A1})$, $y_t \in \mathcal{B}$ and for $r_2 \in]0, r_1[$, there exists $b_1 \in]0, r_1]$ such that $\|y_t - \varphi\|_{\mathcal{B}} \le r_2$ for all $t \in [0, b_1]$.
Set $\overline{M}_a := \sup\limits_{0 \le s \le a} \|S'(s)\|_{\overline{D(A)}}$ and let b be a constant such that

$$0 < b \le \min \left\{ b_1 \, , \, \frac{r_1 - r_2}{\bar{M}\, \overline{M}_a K_a \mu} \right\}.$$

We introduce the space

$$\begin{aligned} \mathbb{F}_b := \quad & \{u : (-\infty, b] \to E \text{ such that } u_0 \in \mathcal{B} \text{ and} \\ & \qquad \text{the restriction } u : [0,b] \to E \text{ is continuous}\}, \end{aligned}$$

endowed with the seminorm $\|\cdot\|_{\mathbb{F}_b}$, defined by

$$\|u\|_{\mathbb{F}_b} := \|u_0\|_{\mathcal{B}} + \sup_{0 \le s \le b} |u(s)| \, .$$

We set also, for $\varphi \in \mathcal{B}$, the following subset of $\mathbb{F}_b$

$$\begin{aligned} \mathbb{F}_b(\varphi) := \quad & \{u \in \mathbb{F}_b \text{ such that } \|u_0 - \varphi\|_{\mathcal{B}} = 0 \text{ and} \\ & \qquad \text{for any } t \in [0,b] \, , \quad \|u_t - \varphi\|_{\mathcal{B}} \le r_1\} \, . \end{aligned}$$

It is clear that the restriction of y to $(-\infty, b]$ is an element of $\mathbb{F}_b(\varphi)$. Then, $\mathbb{F}_b(\varphi)$ is a nonempty.

For any $u \in \mathbb{F}_b(\varphi)$, we have

$$\begin{aligned}
\|u\|_{\mathbb{F}_b} &= \|u_0\|_{\mathcal{B}} + \sup_{0\le s\le b} |u(s)|\,, \\
&\le \|u_0-\varphi\|_{\mathcal{B}} + \|\varphi\|_{\mathcal{B}} + \sup_{0\le s\le b} H\,\|u_s\|_{\mathcal{B}}\,, \\
&\le \|\varphi\|_{\mathcal{B}} + H \sup_{0\le s\le b} \{\|(u_s-\varphi)+\varphi\|_{\mathcal{B}}\}, \\
&\le \|\varphi\|_{\mathcal{B}} + H\{\sup_{0\le s\le b} \|(u_s-\varphi)\|_{\mathcal{B}} + \|\varphi\|_{\mathcal{B}}\}, \\
&\le \|\varphi\|_{\mathcal{B}} + H(r_1 + \|\varphi\|_{\mathcal{B}}).
\end{aligned}$$

Then, $\mathbb{F}_b(\varphi)$ is bounded.
By using the triangular inequality in $\mathcal{B}$, we can clearly see that $\alpha y_1 + (1-\alpha)y_2 \in \mathbb{F}_b(\varphi)$ for any $y_1,\, y_2 \in \mathbb{F}_b(\varphi)$ and $\alpha \in\,]0,1[$. Then, $\mathbb{F}_b(\varphi)$ is convex.

Finally, $\mathbb{F}_b(\varphi)$ is closed in $\mathbb{F}_b$. To prove that, consider a convergent sequence $(u^n)_{n\ge 0}$ of $\mathbb{F}_b(\varphi)$ with $\lim_{n\to+\infty} u^n = u$ in $\mathbb{F}_b$. Then, for any n in $I\!N$, we have

$$\|u_0-\varphi\|_{\mathcal{B}} \le \|u_0-u_0^n\|_{\mathcal{B}} + \|u_0^n-\varphi\|_{\mathcal{B}}\,,$$

letting n to $+\infty$, yields $\|u_0-\varphi\|_{\mathcal{B}} = 0$. In addition, Axiom $(\mathbf{A-iii})$ implies that for any $t\in[0,b]$,

$$\begin{aligned}
\|u_t^n - u_t\|_{\mathcal{B}} &\le K(t) \sup_{0\le s\le t} |u^n(s)-u(s)| + M(t)\,\|u_0^n-u_0\|_{\mathcal{B}} \\
&\le \max(K_a, M_a)\,\|u^n-u\|_{\mathbb{F}_b}\,.
\end{aligned}$$

Then, for any $t\in[0,b]$, $u_t^n \to u_t$ in $\mathcal{B}$. From this together with the following inequality

$$\begin{aligned}
\|u_t-\varphi\|_{\mathcal{B}} &\le \|u_t-u_t^n\|_{\mathcal{B}} + \|u_t^n-\varphi\|_{\mathcal{B}}\,, \\
&\le \|u_t-u_t^n\|_{\mathcal{B}} + r_1, \qquad\qquad \text{for any } n\in I\!N,
\end{aligned}$$

we deduce that $\|u_t-\varphi\|_{\mathcal{B}} \le r_1$. Consequently, $u\in\mathbb{F}_b(\varphi)$.
We have proved that $\mathbb{F}_b(\varphi)$ is a nonempty, bounded, convex and closed subset of $\mathbb{F}_b$.

Consider now the mapping $\mathcal{T}$ defined on $\mathbb{F}_b(\varphi)$ by

$$(\mathcal{T}u)(t) = \begin{cases} S'(t)\varphi(0) + \dfrac{d}{dt}\displaystyle\int_0^t S(t-s)F(s,u_s)ds, & t\in[0,b] \\ \varphi(t), \quad -\infty < t \le 0. \end{cases} \tag{2.2}$$

$\mathcal{T}$ maps elements of $\mathbb{F}_b(\varphi)$ into $\mathbb{F}_b(\varphi)$. In fact, by $(\mathbf{H2.2})$ and Axiom $(\mathbf{A1})$, for every $u\in\mathbb{F}_b(\varphi)$, the mapping $s\mapsto F(s,u_s)$ is continuous on $[0,b]$. So, for every $u\in\mathbb{F}_b(\varphi)$, the mapping $t \mapsto \int_0^t S(t-s)F(s,u_s)ds$ is continuously differentiable on $[0,b]$. From this, the mapping $v := \mathcal{T}u$ is continuous on $[0,b]$. Then, $v \in \mathbb{F}_b$. To prove that $v \in \mathbb{F}_b(\varphi)$, we put $w = v - y$. Then, we get for any $t \in [0,b]$,

$$\begin{aligned} \|v_t - \varphi\|_{\mathcal{B}} &\leq \|w_t\|_{\mathcal{B}} + \|y_t - \varphi\|_{\mathcal{B}} \\ &\leq \|w_t\|_{\mathcal{B}} + r_2. \end{aligned}$$

By Axiom $(\mathbf{A-iii})$, we have for any $t \in [0,b]$,

$$\|w_t\|_{\mathcal{B}} \leq K_a \sup_{0\leq s\leq t} |w(s)|.$$

From Proposition 1.2.9 we obtain for $t \in [0,b]$

$$\begin{aligned} |\lambda R(\lambda,A)w(t)| &= \left| \int_0^t S'(t-s)\lambda R(\lambda,A)F(s,u_s)dt \right| \\ &\leq \frac{\lambda}{\lambda-\omega} \bar{M}\,\overline{M}_a \mu b, \\ &\leq \frac{\lambda}{\lambda-\omega}\left(\frac{r_1-r_2}{K_a}\right). \end{aligned}$$

Letting $\lambda \to +\infty$, we obtain $|w(t)| \leq \dfrac{r_1-r_2}{K_a}$, and $\|v_t-\varphi\|_{\mathcal{B}} \leq r_1$ for any $t \in [0,b]$. Which shows that $v \in \mathbb{F}_b(\varphi)$.

We will prove now the continuity of $\mathcal{T}$. Axioms $(\mathbf{A1})$ and $(\mathbf{A-iii})$ imply that the mapping $G : [0,b]\times\mathbb{F}_b \to \mathcal{B}$, defined by $G(s,u) = u_s$ is continuous. On the other hand, if $(u^n)_{n\geq 1}$ is a convergent sequence in $\mathbb{F}_b(\varphi)$ with $\lim_{n\to+\infty} u^n = u^\infty$, then the set $\{u^\infty\}\bigcup\{u^n : n\geq 1\}$ is compact in $\mathbb{F}_b$. Hence, the set $W = \{(s,u_s^\infty) : 0\leq s\leq b\}\bigcup\{(s,u_s^n) : n\geq 1,\ 0\leq s\leq b\}$ is compact in $[0,b]\times\mathcal{B}$ and F is uniformly continuous in W. Since $(u^n)_{n\geq 1}$ converges to u^∞, (2.2) implies that $(\mathcal{T}u^n)_{n\geq 1}$ converges to $\mathcal{T}u^\infty$.

We will show now that the range of $\mathcal{T}$, $Range(\mathcal{T}) := \{\mathcal{T}u,\ u \in \mathbb{F}_b(\varphi)\}$, is relatively compact in $\mathbb{F}_b(\varphi)$. By Arzéla-Ascoli's theorem, it suffices to prove that $Range(\mathcal{T})(t)$ is relatively compact in E for each $t \in [0,b]$, and $Range(\mathcal{T})$ is equicontinuous on $[0,b]$. To prove the first assertion, it is sufficient to show that the set $\{(\mathcal{T}u)(t) - S'(t)\varphi(0),\quad u \in \mathbb{F}_b(\varphi)\}$ is relatively compact for each $t \in]0,b]$. Let $0<\varepsilon<t$, we have for $\lambda>\omega$

$$
\begin{aligned}
\lambda R(\lambda,A)\frac{d}{dt}\int_0^t & S(t-s)F(s,u_s)ds \\
&= \int_0^t S'(t-s)\lambda R(\lambda,A)F(s,u_s)ds, \\
&= S'(\varepsilon)\int_0^{t-\varepsilon} S'(t-s-\varepsilon)\lambda R(\lambda,A)F(s,u_s)ds \\
&\quad + \int_{t-\varepsilon}^t S'(t-s)\lambda R(\lambda,A)F(s,u_s)ds.
\end{aligned}
$$

Since $S'(\varepsilon)$ is compact, there exists a compact set W_ε such that

$$
\left\{ S'(\varepsilon)\int_0^{t-\varepsilon} S'(t-s-\varepsilon)\lambda R(\lambda,A)F(s,u_s)ds : \ u\in\mathbb{F}_b(\varphi),\ \lambda>\omega \right\} \subseteq W_\varepsilon.
$$

Furthermore,

$$
\left| \int_{t-\varepsilon}^t S'(t-s)\lambda R(\lambda,A)F(s,u_s)\,ds \right| \leq \frac{\lambda}{\lambda-\omega}\bar{M}\,\overline{M}_a\mu\varepsilon, \quad \text{for all } u\in\mathbb{F}_b(\varphi).
$$

This implies, by letting λ to $+\infty$, that $\{(\mathcal{T}u)(t)-S'(t)\varphi(0), \quad u\in\mathbb{F}_b(\varphi)\}$ is totally bounded, that is uniformly bounded for $u\in\mathbb{F}_b(\varphi)$, and therefore $Range(\mathcal{T})(t)$ is relatively compact. On the other hand, for every $0\leq t_0<t\leq b$ and $\lambda>\omega$, one has

$$
\begin{aligned}
\lambda R(\lambda,A)\left((\mathcal{T}u)(t)-(\mathcal{T}u)(t_0)\right) &= \lambda R(\lambda,A)\left(S'(t)-S'(t_0)\right)\varphi(0) \\
&\quad + \int_{t_0}^t S'(t-s)\lambda R(\lambda,A)F(s,u_s)ds \\
&\quad + \int_0^{t_0} \left(S'(t-s)-S'(t_0-s)\right)\lambda R(\lambda,A)F(s,u_s)ds.
\end{aligned}
$$

This leads to

$$
\begin{aligned}
&|\lambda R(\lambda,A)\left((\mathcal{T}u)(t)-(\mathcal{T}u)(t_0)\right)| \\
&\quad \leq |\lambda R(\lambda,A)\left(S'(t)-S'(t_0)\right)\varphi(0)| + \frac{\lambda}{\lambda-\omega}\bar{M}\,\overline{M}_a\mu(t-t_0) \\
&\quad + \left|\left(S'(t-t_0)-I\right)\int_0^{t_0} S'(t_0-s)\lambda R(\lambda,A)F(s,u_s)ds\right|.
\end{aligned} \tag{2.3}
$$

Besides, there exists a compact set W such that

$$
\left\{ \int_0^{t_0} S'(t_0-s)\lambda R(\lambda,A)F(s,u_s)ds : \quad u\in\mathbb{F}_b(\varphi),\ \lambda>\omega \right\} \subseteq W.
$$

Letting λ to $+\infty$ and using the equicontinuity of $(S'(\cdot)x)_{x\in W}$, we obtain

$$\lim_{t\to t_0} |(\mathcal{T}u)(t) - (\mathcal{T}u)(t_0)| = 0. \tag{2.4}$$

Using similar argument for $0 \leq t < t_0 \leq b$, we can conclude that $Range(\mathcal{T})(t)$ is equicontinuous.

Finally, the Schauder fixed point theorem implies that $\mathcal{T}$ has a fixed point x in $\mathbb{F}_b(\varphi)$. The fact that x is an integral solution of Equation (2.1) is a consequence of the same property in Theorem 1.2.8. This ends the proof of Theorem 2.2.3. ■

In order to define the integral solution to Equation (2.1) in its maximal interval of existence, we add the following condition:

(**H2.3**) F is continuous from $[0,+\infty) \times \mathcal{B}$ into E and takes bounded sets of $[0,+\infty) \times \mathcal{B}$ into bounded sets of E.

Theorem 2.2.4 *Assume that the conditions* (**H2.1**), (**H2.2**) *and* (**H2.3**) *hold and let* $\varphi \in \mathcal{B}$ *such that* $\varphi(0) \in \overline{D(A)}$. *Then, Equation (2.1) has at least one integral solution* $x(.,\varphi)$ *in a maximal interval* $(-\infty, b_\varphi)$. *Moreover,*

i) $b_\varphi = +\infty$ *or*

ii) $b_\varphi < +\infty$ *and* $\limsup_{t\to b_\varphi^-} |x(t,\varphi)| = +\infty.$

Proof. Let $x(.,\varphi)$ be an integral solution of Equation (2.1) in $(-\infty, b_1]$. We know that $x(t) \in \overline{D(A)}$, for all $t \in [0, b_1]$. Repeating the procedure used in the local existence result. This yields existence of $b_2 > b_1$ and a function $x(., x_{b_1}(.,\varphi)) : (-\infty, b_2] \to E$ which satisfies for $t \in [b_1, b_2]$

$$x(t, x_{b_1}(.,\varphi)) = S'(t-b_1)x(b_1,\varphi) + \frac{d}{dt}\int_{b_1}^{t} S(t-s)F(s, x_s(., x_{b_1}(.,\varphi)))ds.$$

Proceeding inductively, we obtain the maximal interval of existence $(-\infty, b_\varphi)$ of the solution $x(.,\varphi)$. Assume that $b_\varphi < \infty$ and $\limsup_{t\to b_\varphi} |x(t,\varphi)|$ is bounded. Then, from (**H2.3**) there exists a constant $\mu > 0$ such that $|F(t, x_t(.,\varphi)| \leq \mu$ for all $t \in [0, b_\varphi)$. As before, let $\overline{M}_{b_\varphi} = \sup_{0\leq s\leq b_\varphi} \|S'(s)\|_{\overline{D(A)}}$ and $x : [t_0, b_\varphi) \to E$, $t_0 \in (0, b_\varphi)$, be the restriction of $x(.,\varphi)$ to $[t_0, b_\varphi)$. Consider $t_0 \leq t \leq t' < b_\varphi$ and $0 < \varepsilon < t_0$. Let $\eta_1 > 0$ be such that $|(S'(t') - S'(t))\,\varphi(0)| \leq \varepsilon$ for $|t-t'| < \eta_1$. For $\lambda > \omega$, one has

$$
\begin{aligned}
|\lambda R(\lambda, A)\,(x(t') - x(t))| \le{}& |\lambda R(\lambda, A)\,(S'(t') - S'(t))\,\varphi(0)| \\
&+ \int_{t-\varepsilon}^{t} |(S'(t'-s) - S'(t-s))\,\lambda R(\lambda, A) F(s, x_s)|\,ds \\
&+ \int_{t}^{t'} S'(t'-s)\lambda R(\lambda, A) F(s, x_s) ds \\
&+ \int_{0}^{t-\varepsilon} |(S'(t'-\varepsilon-s) - S'(t-\varepsilon-s))\,S'(\varepsilon)\lambda R(\lambda, A) F(s, x_s)|\,ds.
\end{aligned}
$$

Moreover, $S'(\varepsilon)\,\{\lambda R(\lambda, A) F(s, x_s) :\ \lambda > \omega\}$ is included in a compact set W. Since, the family $(S'(\cdot)x)_{x\in W}$ is uniformly equicontinuous in $[0, b_\varphi - \varepsilon]$, there exists $\eta_2 > 0$ such that

$$
\left|\left(S'(t') - S'(t)\right) u\right| < \varepsilon, \ \text{ for } u \in W,
$$

whenever $t, t' \in [0, b_\varphi - \varepsilon]$ and $|t - t'| < \eta_2$.
Consequently, if $|t' - t| < \inf(\eta_1, \eta_2, \varepsilon)$, then

$$
\left|\lambda R(\lambda, A)\,\left(x(t') - x(t)\right)\right| \le \frac{\lambda}{\lambda - \omega} \bar{M}\,\left(\varepsilon + 3\overline{M}_{b_\varphi}\varepsilon\mu\right) + (t - \varepsilon)\varepsilon.
$$

By letting λ to $+\infty$, we obtain

$$
\left|x(t') - x(t)\right| \le (\bar{M} + 3\bar{M}\,\overline{M}_{b_\varphi}\mu + b_\varphi)\varepsilon.
$$

Using a similar argument for $t_0 \le t' \le t < b_\varphi$, we can conclude that

$$
\lim_{|t-t'|\to 0} \left|x(t') - x(t)\right| = 0.
$$

Therefore, $\lim\limits_{t\to b_\varphi} x(t, \varphi)$ exists. If we define $x(b_\varphi, \varphi) := \lim\limits_{t\to b_\varphi} x(t, \varphi)$, we can extend $x(., \varphi)$ beyond b_φ and we contradict the maximality of $(-\infty, b_\varphi)$. This finishes the proof. ■

2.2.2 Global existence and uniqueness of integral solutions

Our objective here is to give sufficient conditions for global existence and uniqueness of integral solutions to Equation (3.2). We keep the assumption (**H2.1**) and instead of the assumptions (**H2.2**) and (**H2.3**), we make the following condition :
(**H2.4**) $F : [0, +\infty) \times \mathcal{B} \to E$ is continuous and satisfies the Lipschitz condition : for each $a > 0$ there exists a positive constant L such that $|F(t, \varphi) - F(t, \psi)| \le L\,\|\varphi - \psi\|_{\mathcal{B}}$, for all $t \in [0, a]$ and $\varphi, \psi \in \mathcal{B}$.

Theorem 2.2.5 *Assume that the conditions* (**H2**.1) *and* (**H2**.4) *are satisfied. Then, for* $\varphi \in \mathcal{B}$ *such that* $\varphi(0) \in \overline{D(A)}$, *Equation (2.1) has a unique global integral solution* x *which is given by*

$$x(t) = \begin{cases} S'(t)\varphi(0) + \dfrac{d}{dt}\displaystyle\int_0^t S(t-s)F(s,x_s)ds, & t \geq 0, \\ \varphi(t), & -\infty < t \leq 0. \end{cases} \tag{2.5}$$

Moreover, $x(t,\varphi)$ *is a continuous function of* φ*, in the sense that if* $\varphi \in \mathcal{B}$ *such that* $\varphi(0) \in \overline{D(A)}$ *and* $t \geq 0$*, then there exists a positive constant* β *such that, for* $\psi \in \mathcal{B}$ *and* $\psi(0) \in \overline{D(A)}$*, we have*

$$|x(t,\varphi) - x(t,\psi)| \leq \beta \|\varphi - \psi\|_{\mathcal{B}}.$$

Proof. Let $a > 0$ be fixed and $\varphi \in \mathcal{B}$ such that $\varphi(0) \in \overline{D(A)}$. Consider the set $Z_a(\varphi) := \{y \in C([0,a];E) : \ y(0) = \varphi(0)\}$ and $z \in Z_a(\varphi)$.
Let $\tilde{z} : (-\infty, a] \to E$ be the mapping defined by

$$\tilde{z}(t) = \begin{cases} z(t), & t \in [0,a], \\ \varphi(t), & -\infty < t \leq 0. \end{cases}$$

By virtue of condition (**H2**.4) and Axiom (**A1**) the mapping $s \mapsto F(s,\tilde{z}_s)$ is continuous on $[0,a]$. This implies that the mapping $t \mapsto \int_0^t S(t-s)F(s,\tilde{z}_s)\,ds$ is continuously differentiable on $[0,a]$.
Consider the operator $P : Z_a(\varphi) \longrightarrow Z_a(\varphi)$ defined by

$$(Pz)(t) := S'(t)\varphi(0) + \frac{d}{dt}\int_0^t S(t-s)F(s,\tilde{z}_s)ds.$$

Thanks to Proposition 1.2.5, we can assume without loss of generality that $\omega > 0$. By virtue of Proposition 1.2.9, we have for every $y, z \in Z_a(\varphi)$, $t \in [0,a]$ and $\lambda > \omega$,

$$\lambda R(\lambda, A)\left((Py)(t) - (Pz)(t)\right) = \int_0^t S'(t-s)\lambda R(\lambda,A)\left(F(s,\tilde{y}_s) - F(s,\tilde{z}_s)\right)ds.$$

Then,

$$|\lambda R(\lambda, A)\left((Py)(t) - (Pz)(t)\right)| \leq \frac{\lambda}{\lambda - \omega}\bar{M}^2 e^{\omega a}\int_0^t |F(s,\tilde{y}_s) - F(s,\tilde{z}_s)|\,ds.$$

Thus, from (**H2**.4) and by letting λ to $+\infty$, we obtain

$$|(Py)(t) - (Pz)(t)| \leq L\bar{M}^2 e^{\omega a}\int_0^t \|\tilde{y}_s - \tilde{z}_s\|_{\mathcal{B}}\,ds.$$

This yields

$$|(Py)(t)-(Pz)(t)| \leq L\bar{M}^2 e^{\omega a}\int_0^t K(s)\sup_{0\leq\xi\leq s}|y(\xi)-z(\xi)|\,ds,$$
$$\leq L\bar{M}^2 K_a e^{\omega a}\int_0^t \sup_{0\leq\xi\leq s}|y(\xi)-z(\xi)|\,ds.$$

Following the same reasoning, we can see that

$$\begin{aligned}\left|(P^2y)(t)-(P^2z)(t)\right| &\leq L\bar{M}^2 K_a e^{\omega a}\int_0^t \sup_{0\leq\xi\leq s}|(Py)(\xi)-(Pz)(\xi)|\,ds,\\ &\leq (L\bar{M}^2 K_a e^{\omega a})^2\int_0^t \sup_{0\leq\xi\leq s}\int_0^\xi \sup_{0\leq\alpha\leq p}|y(\alpha)-z(\alpha)|\,dp\,ds,\\ &\leq (L\bar{M}^2 K_a e^{\omega a})^2\int_0^t\int_0^s \|y-z\|_{Z_a(\varphi)}\,dp\,ds,\\ &\leq \frac{(L\bar{M}^2 K_a e^{\omega a})^2}{2}a^2\|y-z\|_{Z_a(\varphi)}.\end{aligned}$$

We can repeat the previous argument, and we obtain

$$|(P^ny)(t)-(P^nz)(t)| \leq \frac{(L\bar{M}^2 K_a e^{\omega a})^n}{n!}a^n\|y-z\|_{Z_a(\varphi)}.$$

Since there exists $m\in I\!N$ such that $\dfrac{(L\bar{M}^2 K_a e^{\omega a})^m}{m!}a^m<1$, it follows that P^m is a strict contraction on the closed subset $Z_a(\varphi)$ of the Banach space $C([0,a];E)$. Consequently, by the Banach fixed point theorem, we deduce the existence and uniqueness of $x:=x(.,\varphi)\in Z_a(\varphi)$ such that $Px=x$. Finally, for all $a>0$, Equation (2.5) has a unique solution which is defined on the interval $(-\infty,a]$. Then, x is a global solution of Equation (2.5).
On the other hand, if we consider two solutions $x:=x(.,\varphi)$ and $y:=y(.,\psi)$ for $\varphi\in\mathcal{B}$ with $\varphi(0)\in\overline{D(A)}$ and $\psi\in\mathcal{B}$ with $\psi(0)\in\overline{D(A)}$, then for every $t\in[0,a]$, with $a>0$ fixed

$$|x(t)-y(t)| \leq |S'(t)\varphi(0)-S'(t)\psi(0)|+L\bar{M}^2 e^{\omega a}\int_0^t \|\tilde{x}_s-\tilde{y}_s\|_{\mathcal{B}}\,ds,$$

$$\begin{aligned}&\leq \bar{M}e^{\omega a}|\varphi(0)-\psi(0)|\\ &\quad +L\bar{M}^2 e^{\omega a}\int_0^t\left(K(s)\max_{0\leq\xi\leq s}|x(\xi)-y(\xi)|+M(t)\|\varphi-\psi\|_{\mathcal{B}}\right)ds,\end{aligned}$$

$$\begin{aligned}&\leq H\bar{M}e^{\omega a}\|\varphi-\psi\|_{\mathcal{B}}\\ &\quad +L\bar{M}^2 K_a e^{\omega a}\int_0^t \max_{0\leq\xi\leq s}|x(\xi)-y(\xi)|\,ds+aL\bar{M}^2 M_a e^{\omega a}\|\varphi-\psi\|_{\mathcal{B}}.\end{aligned}$$

By Gronwall's lemma, it follows that

$$|x(t) - y(t)| \leq \beta \|\varphi - \psi\|_{\mathcal{B}}, \quad \text{for } t \in [0, a],$$

where

$$\beta = \bar{M} e^{\omega a}(H + aL\bar{M}M_a) \exp(aL\bar{M}^2 K_a e^{\omega a}).$$

Hence the continuity of $x(t, \varphi)$ in term of φ and the uniqueness of the solution of Equation (2.5) are guaranteed. This completes the proof. ■

2.2.3 Existence of strict solutions

Under more restrictive conditions, we obtain existence of strict solutions to Equation (2.1). In order to compute the integral in $\mathcal{B}$ from the integral in E, we suppose that $\mathcal{B}$ is normed and satisfies one of the following two extra axioms.

(**C1**) If $(\phi_n)_{n\geq 0}$ is a Cauchy sequence in $\mathcal{B}$ with respect to the seminorm and if $(\phi_n)_{n\geq 0}$ converges compactly to ϕ on $(-\infty, 0]$, then ϕ is in $\mathcal{B}$ and $\|\phi_n - \phi\|_{\mathcal{B}} \to 0$, as $n \to \infty$.

(**D1**) For a sequence $(\varphi_n)_{n\geq 0}$ in $\mathcal{B}$, if $\|\varphi_n\|_{\mathcal{B}} \to 0$, as $n \to \infty$, then $|\varphi_n(\theta)| \to 0$, as $n \to \infty$, for each $\theta \in (-\infty, 0]$.

Lemma 2.2.6 *[92] Let $\mathcal{B}$ be a normed space which satisfies Axiom* (**C1**) *and $f : [0, a] \to \mathcal{B}$, $a > 0$, be a continuous function such that $f(t)(\theta)$ is continuous for $(t, \theta) \in [0, a] \times (-\infty, 0]$.*

Then, $$\left[\int_0^a f(t)dt\right](\theta) = \int_0^a f(t)(\theta)dt, \qquad \theta \in (-\infty, 0].$$

We can obtain a similar result by using Axiom (**D1**).

Lemma 2.2.7 *Let $\mathcal{B}$ satisfies Axiom* (**D1**) *and $f : [0, a] \to \mathcal{B}$ be a continuous function. Then for all $\theta \in (-\infty, 0]$, the function $f(.)(\theta)$ is continuous and*

$$\left[\int_0^a f(t)dt\right](\theta) = \int_0^a f(t)(\theta)dt, \qquad \theta \in (-\infty, 0].$$

Proof. We have $\int_0^a f(t)dt = \lim_{n\to+\infty} \frac{a}{n} \sum_{k=1}^{n} f(\frac{ka}{n})$ in $\mathcal{B}$. By use of Axiom (**D1**), we get

$$\left[\int_0^a f(t)dt\right](\theta) = \lim_{n\to+\infty} \frac{a}{n} \sum_{k=1}^{n} f(\frac{kn}{a})(\theta), \qquad \theta \in (-\infty, 0].$$

On the other hand, the same axiom implies that the function $f(.)(\theta)$ is continuous on $[0, a]$. From what we infer that for all $\theta \in (-\infty, 0]$, the function $f(.)(\theta)$ is integrable on $[0, a]$ and

$$\int_0^a f(t)(\theta)dt = \lim_{n\to+\infty} \frac{a}{n} \sum_{k=1}^{n} f(\frac{kn}{a})(\theta), \qquad \theta \in (-\infty, 0].$$

This ends the proof of the lemma. ■

To prove the strictness of solutions, we need also to make the following assumption.
(**H2.5**) $F : [0, \infty) \times \mathcal{B} \to E$ is continuously differentiable and the derivatives D_1F, D_2F satisfy the following locally Lipschitz condition :
for any compact set $Q \subset [0, \infty) \times \mathcal{B}$, there exists a constant $L > 0$ such that

$$\begin{cases} \|D_1F(t,\varphi) - D_1F(t,\psi)\| \leq L \|\varphi - \psi\|_{\mathcal{B}}, \\ \|D_2F(t,\varphi) - D_2F(t,\psi)\| \leq L \|\varphi - \psi\|_{\mathcal{B}}, \end{cases}$$

for all $(t,\varphi), (t,\psi) \in Q$.

Theorem 2.2.8 *Assume that $\mathcal{B}$ is a normed space which satisfies Axiom* (**C1**) *or Axiom* (**D1**) *and the conditions* (**H2.1**), (**H2.4**) *and* (**H2.5**) *hold. Let $\varphi \in \mathcal{B}$ continuously differentiable such that*

$$\varphi' \in \mathcal{B}, \ \varphi(0) \in D(A), \ \varphi'(0) \in \overline{D(A)} \text{ and } \varphi'(0) = A\varphi(0) + F(0, \varphi).$$

Then, the integral solution of Equation (2.1) is a strict solution.

Proof. By Theorem 2.2.5, we know that Equation (2.1) has a unique global integral solution x, which is given by

$$x(t) = S'(t)\varphi(0) + \frac{d}{dt}\int_0^t S(t-s)F(s, x_s)ds, \quad \text{for } t \geq 0. \tag{2.6}$$

The assumption $\varphi(0) \in D(A)$ implies that $S'(t)\varphi(0) = S(t)A\varphi(0) + \varphi(0)$, and x can be written as

$$x(t) = S(t)A\varphi(0) + \varphi(0) + \frac{d}{dt}\int_0^t S(t-s)F(s, x_s)ds. \tag{2.7}$$

Consider the following equation

$$\begin{cases} \dfrac{d}{dt}y(t) = Ay(t) + D_1F(t, x_t) + D_2F(t, x_t)\, y_t, & t \geq 0, \\ y(t) = \varphi'(t), \quad -\infty < t \leq 0. \end{cases} \tag{2.8}$$

Using Axiom (**A1**)**,** we can see that the mapping $G : (-\infty, 0] \times \mathcal{B} \to E$ defined by $G(t,\psi) = D_1F(t, x_t) + D_2F(t, x_t)\psi$, satisfies (**H2.4**). By Theorem 2.2.5, Equation (2.8) has a unique global integral solution y which is given by

$$\begin{cases} y(t) = S'(t)(A\varphi(0) + F(0,\varphi)) \\ \qquad + \dfrac{d}{dt}\displaystyle\int_0^t S(t-s)\left(D_1F(s, x_s) + D_2F(s, x_s)\, y_s\right) ds, \quad t \geq 0, \\ \\ y(t) = \varphi'(t), \quad -\infty < t \leq 0. \end{cases} \tag{2.9}$$

Consider the mapping $w : (-\infty, +\infty) \to E$, defined by

$$w(t) = \begin{cases} \varphi(0) + \displaystyle\int_0^t y(s)\,ds, & t \geq 0, \\ \varphi(t), & -\infty < t \leq 0. \end{cases} \tag{2.10}$$

We will show that $x = w$. We have

$$\begin{cases} w(t) = S(t)(A\varphi(0) + F(0,\varphi)) + \varphi(0) \\ \qquad\qquad + \displaystyle\int_0^t S(t-s)\left(D_1F(s,x_s) + D_2F(s,x_s)\,y_s\right)ds, \quad t \geq 0, \\ \\ w(t) = \varphi(t), \quad -\infty < t \leq 0. \end{cases} \tag{2.11}$$

Lemma 2.2.6 or Lemma 2.2.7 implies that $w_t = \varphi + \displaystyle\int_0^t y_s\,ds$. Then, the mappings $t \mapsto w_t$ and $t \mapsto \displaystyle\int_0^t S(t-s)F(s,w_s)ds$ are continuously differentiable and satisfy

$$\begin{aligned} \frac{d}{dt}\int_0^t S(t-s)F(s,w_s)ds &= S(t)F(0,\varphi) \\ &\quad + \int_0^t S(t-s)\left(D_1F(s,w_s) + D_2F(s,w_s)\,y_s\right)ds. \end{aligned} \tag{2.12}$$

From (2.7), (2.11) and (2.12), we obtain

$$\begin{aligned} x(t) - w(t) &= \frac{d}{dt}\int_0^t S(t-s)\left(F(s,x_s) - F(s,w_s)\right)ds \\ &\quad - \int_0^t S(t-s)\left(D_1F(s,x_s) - D_1F(s,w_s)\right)ds \\ &\quad - \int_0^t S(t-s)\left(D_2F(s,x_s) - D_2F(s,w_s)\right)y_s\,ds. \end{aligned}$$

Let

$$\begin{cases} I_1 = \dfrac{d}{dt}\displaystyle\int_0^t S(t-s)\left(F(s,x_s) - F(s,w_s)\right)ds, \\ I_2 = -\displaystyle\int_0^t S(t-s)\left(D_1F(s,x_s) - D_1F(s,w_s)\right)ds, \\ I_3 = -\displaystyle\int_0^t S(t-s)\left(D_2F(s,x_s) - D_2F(s,w_s)\right)y_s\,ds, \end{cases}$$

and, for $a > 0$ fixed

$$b_0 = \max\left\{\sup_{0\leq s\leq a}\|S(s)\|\,;\ \sup_{0\leq s\leq a}\left\|S'(s)\right\|\,;\ \sup_{0\leq s\leq a}\left(K(s)\,|y(s)| + M(s)\left\|\varphi'\right\|_{\mathcal{B}}\right)\right\}.$$

Let $t \in [0,a]$. It's easy to see that

$$\|I_2\| \le b_0 L \int_0^t \|x_s - w_s\|_{\mathcal{B}}\, ds,$$

and by Axiom $(\mathbf{A} - \mathbf{iii})$,

$$\|I_3\| \le b_0^2 L \int_0^t \|x_s - w_s\|_{\mathcal{B}}\, ds.$$

Proceeding as in the previous proofs, we get

$$\|I_1\| \le b_0 \bar{M} L \int_0^t \|x_s - w_s\|_{\mathcal{B}}\, ds.$$

Let

$$b_1 = b_0 L \left(1 + b_0 + \bar{M}\right).$$

Then, we obtain

$$\sup_{0 \le s \le t} |x(s) - w(s)| \le b_1 \int_0^t \|x_s - w_s\|_{\mathcal{B}}\, ds$$

Since $x_0 = w_0 = \varphi$, Axiom $(\mathbf{A} - \mathbf{iii})$ implies that

$$\|x_t - w_t\|_{\mathcal{B}} \le K_a \sup_{0 \le s \le t} |x(s) - w(s)|.$$

Then,

$$\|x_t - w_t\|_{\mathcal{B}} \le K_a b_1 \int_0^t \|x_s - w_s\|_{\mathcal{B}}\, ds.$$

Using Gronwall's lemma, we conclude that

$$\|x_t - w_t\|_{\mathcal{B}} = 0 \text{ for any } t \in [0,a].$$

Consequently, $x(t) = w(t)$ for all $t \in (-\infty, a]$. Hence, x is continuously differentiable on $(-\infty, a]$. This ends the proof. ■

2.3 An application to partial integrodifferential equations with infinite delay

In this section, we make an attempt to apply some of the results obtained in the previous sections. We consider the following partial integrodifferential equation :

$$
\begin{cases}
\dfrac{\partial}{\partial t} w(t,\xi) = d\Delta w(t,\xi) + \displaystyle\int_{-\infty}^{t} k(t,s,w(s,\xi))ds + f(t,w(t,\xi)), \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \geq 0, \ \xi \in \quad , \\
w(t,\xi) = 0, \quad t \geq 0, \quad \xi \in \partial \quad , \\
w(\theta,\xi) = w_0(\theta,\xi), \quad -\infty < \theta \leq 0, \quad \xi \in \overline{\ \ }\ ,
\end{cases}
\tag{2.13}
$$

where $d > 0$, is a bounded open set in $I\!R^n$ with regular boundary ∂ , $\Delta = \sum\limits_{i=1}^{n} \dfrac{\partial^2}{\partial \zeta_i^2}$ and $k : I\!R_+ \times I\!R \times I\!R^n \to I\!R^n$; $f : I\!R_+ \times I\!R^n \to I\!R^n$ and $w_0 : (-\infty, 0] \times \overline{\ \ } \to I\!R^n$ are functions.

We choose $E = C\left(\overline{\ \ }, I\!R^n\right)$, and we consider the operator $A : D(A) \subset E \to E$ defined by

$$
\begin{cases}
D(A) = \left\{ y \in E : \ \Delta y \in E \ \text{ and } \ y_{/\partial} \ \ = 0 \right\} \\
Ay = d\Delta y.
\end{cases}
$$

It is known that $\overline{D(A)} = \left\{ y \in E : \ y_{/\partial} \ \ = 0 \right\} \neq E$ and

$$
\begin{cases}
\rho(A) = (0, +\infty), \\
\left\| (\lambda I - A)^{-1} \right\| \leq \dfrac{1}{\lambda}, \qquad \text{for } \lambda > 0.
\end{cases}
$$

This implies that A satisfies (**H2.1**) on E.

As a concrete phase space, we choose the space

$$
\mathcal{B} = C_g^0 := \{ \phi \in C((-\infty, 0]; E) : \lim_{\theta \to -\infty} \frac{|\phi(\theta)|}{g(\theta)} = 0 \},
$$

endowed with the following norm

$$
\|\phi\|_{\mathcal{B}} := \sup_{-\infty < \theta \leq 0} \frac{|\phi(\theta)|}{g(\theta)},
$$

where $g : (-\infty, 0] \to (0, +\infty)$ is a continuous function satisfying

(g1) $g(0) = 1$.

(g2) The function $G : [0, +\infty) \to [0, +\infty)$ defined by

$$
G(t) = \sup_{-\infty < \theta \leq -t} \frac{g(t+\theta)}{g(\theta)},
$$

is locally bounded.

From Theorems 1.**3.2** and 1.**3.6** of [66], this space satisfies Axioms (**A**), (**A1**), (**B**) and (**C1**).

In addition, we set for $t \geq 0$ and $\xi \in \overline{\;\;}$

$$\begin{cases} x(t)(\xi) = w(t,\xi), \\ \varphi(\theta)(\xi) = w_0(\theta,\xi), \\ F(t,\phi)(\xi) = f(t,\phi(0)(\xi)) + \displaystyle\int_{-\infty}^{0} k(t,t+\theta,\phi(\theta)(\xi))\, d\theta, \qquad \text{for } \phi \in \mathcal{B}. \end{cases}$$

Now it will be easy to adapt our previous results to solving Equation (2.13). The properties of the mapping F depend on k, f and on the choice of the space $\mathcal{B}$.

We assume that $k : I\!R_+ \times I\!R \times I\!R^n \to I\!R^n$ and $f : I\!R_+ \times I\!R^n \to I\!R^n$ are continuous functions that satisfy:

(**a**) $\|k(t,s,v(\cdot))\| \leq C(s-t)\,|v|\,, \quad s \leq t\,, \quad v \in E;$

(**b**) $\|k(t,s,u(\cdot)) - k(t,s,v(\cdot))\| \leq N(s-t)\,|u-v|\,, \quad s \leq t, \quad u,v \in E;$

(**c**) $\|f(t,u(\cdot)) - f(t,v(\cdot))\| \leq \eta\,|u-v|\,, \quad t \geq 0, \quad u,v \in E;$

where η is a fixed positive constant, C, $N : (-\infty,0] \to [0,+\infty)$ are two measurable functions such that, $C(\cdot)g(\cdot)$ and $N(\cdot)g(\cdot)$ are integrable on $(-\infty,0]$.

Under the above conditions, $F : [0,+\infty) \times \mathcal{B} \to E$ satisfies condition (**H2.4**). In fact, given $t \geq 0$, $\phi \in \mathcal{B}$ and a sequence $(t_n)_{n\geq 0}$ of $[0,+\infty)$ such that $t_n \to t$; we have

$$\begin{aligned} |F(t_n,\phi) - F(t,\phi)| \leq & \sup_{\xi\in\overline{\;\;}} |f(t_n,\phi(0)(\xi)) - f(t,\phi(0)(\xi))| \\ & + \sup_{\xi\in\overline{\;\;}} \left(\int_{-\infty}^{0} |k(t_n,t_n+\theta,\phi(\theta)(\xi)) - k(t,t+\theta,\phi(\theta)(\xi))|\, d\theta \right), \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} \|k(t_n,t_n+\theta,\phi(\theta)(\cdot))\| & \leq C(\theta)\,|\phi(\theta)|\,, \\ & \leq C(\theta)g(\theta) \sup_{-\infty<\theta\leq 0} \frac{|\phi(\theta)|}{g(\theta)}, \\ & = C(\theta)g(\theta)\,\|\phi\|_{\mathcal{B}}\,. \end{aligned}$$

By continuity of k, we have

$$\lim_{n} k(t_n,t_n+\theta,\phi(\theta)(\xi)) = k(t,t+\theta,\phi(\theta)(\xi)), \text{ uniformely on } \xi \in \overline{\;\;}.$$

Then, the Lebesgue convergence theorem allows us to assert that

$$\sup_{\xi\in\overline{\;\;}}\left(\int_{-\infty}^{0}|k(t_n,t_n+\theta,\phi(\theta)(\xi))-k(t,t+\theta,\phi(\theta)(\xi))|\,d\theta\right)\underset{n}{\rightarrow}0,$$

and by the continuity of f, we obtain

$$\sup_{\xi\in\overline{\;\;}}|f(t_n,\phi(0)(\xi))-f(t,\phi(0)(\xi))|\underset{n}{\rightarrow}0.$$

Then, F is continuous in $t\in[0,+\infty)$.

Furthermore, by Axiom $(\mathbf{A}-\mathbf{ii})$, we have for every ϕ, $\psi\in\mathcal{B}$ and $t\geq 0$

$$\begin{aligned}\|f(t,\phi(0)(\cdot))-f(t,\psi(0)(\cdot))\| &\leq \eta\,|\phi(0)-\psi(0)|\,,\\ &\leq \eta H\,\|\phi-\psi\|_{\mathcal{B}}\,,\end{aligned}$$

and

$$\int_{-\infty}^{0}\|k(t,t+\theta,\phi(\theta)(\cdot))-k(t,t+\theta,\psi(\theta)(\cdot))\|\,d\theta\leq\|\phi-\psi\|_{\mathcal{B}}\int_{-\infty}^{0}N(\theta)g(\theta)d\theta.$$

This implies that

$$|F(t,\phi)-F(t,\psi)|\leq L\,\|\phi-\psi\|_{\mathcal{B}}\ ,$$

where

$$L:=\eta H+\int_{-\infty}^{0}N(\theta)g(\theta)d\theta.$$

Hence, $(\mathbf{H2.4})$ is satisfied.

Assume also that

i) $w_0\in C((-\infty,0]\times\overline{\;\;};\,I\!R^n)$, with $\displaystyle\lim_{\theta\to-\infty}\frac{|w_0(\theta,.)|}{g(\theta)}=0$;

ii) $w_0(0,\xi)=0$, for $\xi\in\partial\;\;$.

Then all conditions of Theorem 2.2.5 are satisfied, and Equation (2.13) has a unique global integral solution on $(-\infty,+\infty)$.

Under more restrictive conditions, we obtain the strict solution.

iii) $w_0\in C^2((-\infty,0]\times\overline{\;\;};\,I\!R^n)$, with $\displaystyle\lim_{\theta\to-\infty}\frac{1}{g(\theta)}\left|\frac{\partial}{\partial\theta}w_0(\theta,.)\right|=0$;

iv) $\dfrac{\partial}{\partial\theta}w_0(0,\xi)=0$, for $\xi\in\partial\;\;$;

v) $\dfrac{\partial}{\partial\theta}w_0(0,\xi)=\Delta w_0(0,\xi)+f(0,w_0(0,\xi))+\displaystyle\int_{-\infty}^{0}k(0,\theta,w_0(\theta,\xi))d\theta$,
for $\xi\in\overline{\;\;}$;

vi) f and k are continuously differentiable, with the following conditions:

(**a1**) $\|D_1 k(t,s,v(\cdot))\| \leq C_1(s-t)\,|v|, \quad \text{for} \;\; s \leq t, \quad v \in E;$

(**b1**) $\|D_1 k(t,s,u(\cdot)) - D_1 k(t,s,v(\cdot))\| + \|D_2 k(t,s,u(\cdot)) - D_2 k(t,s,v(\cdot))\|$
$\leq N_1(s-t)\,|u-v|\,, \quad \text{for} \;\; s \leq t, \quad u, v \in E;$

(**c1**) $\|D_1 f(t,u(\cdot)) - D_1 f(t,v(\cdot))\| \leq \eta_1\,|u-v|\,, \quad \text{for} \;\; t \geq 0, \quad u, v \in E;$

(**a2**) $\|D_3 k(t,s,v(\cdot))\| \leq C_2(s-t)\,|v|\,, \quad \text{for} \;\; s \leq t, \quad v \in E;$

(**b2**) $\|D_3 k(t,s,u(\cdot)) - D_3 k(t,s,v(\cdot))\| \leq N_2(s-t)\,|u-v|\,,$
for $s \leq t, \quad u, v \in E;$

(**c2**) $\|D_2 f(t,u(\cdot)) - D_2 f(t,v(\cdot))\| \leq \eta_2\,|u-v|\,, \quad \text{for} \;\; t \geq 0, \quad u, v \in E;$

where D_1, D_2 and D_3 are the derivatives, η_1 and η_2 are two fixed positive constants, C_1, N_1, C_2, $N_2 : (-\infty, 0] \to [0, +\infty)$ are measurable functions such that $C_1(\cdot)g(\cdot)$, $N_1(\cdot)g(\cdot)$, $C_2(\cdot)g(\cdot)$ and $N_2(\cdot)g(\cdot)$ are integrable on $(-\infty, 0]$. We can verify without difficulties, that all conditions of Theorem 2.2.8 hold. Then the integral solution $w(t,\xi) = x(t)(\xi)$ is a strict one.

Chapter 3

Local Existence, Stability and Attractiveness for Some Partial Functional Differential Equations with Infinite Delay[1]

3.1 Introduction

This chapter is concerned with local existence of solutions and stability to the autonomous case of the PFDE with infinite delay considered in the previous chapter

$$\begin{cases} \frac{d}{dt}x(t) = Ax(t) + F(x_t), \quad t \geq 0, \\ x_0 = \varphi \in \mathcal{B}, \end{cases} \tag{3.1}$$

with the same notations as before.

Under different assumptions, we had proved in Chapter 2 that the density condition $\overline{D(A)} = E$, is not necessary to deal with existence of solutions to PFDEs with infinite delay. Here, first, we state the local existence and regularity of solutions under a local Lipschitz condition on F. Then, in the case of global existence, we give some properties of the solution map. Next, we investigate the stability near an equilibrium. Mainly, we prove that if the linearized

[1] This chapter is based on two papers. The first one is in collaboration with M. Adimy and K. Ezzinbi and it will appear in Journal of Nonlinear Analysis, Theory, Methods and Applications, (2001). The second one is in collaboration with K. Ezzinbi and it will appear in Fields Institue Communications Series, (2001).

semigroup around an equilibrium is exponentially stable, then the equilibrium of Equation (3.2) is also exponentially stable. After that, we deal with existence of a global attractor. Finally, we give an application to a reaction diffusion equation with infinite delay

We still assume that the phase space $(\mathcal{B}, \|.\|_{\mathcal{B}})$ is a seminormed linear space of functions mapping $(-\infty, 0]$ into E, that satisfies the fundamental Axioms (**A**), (**A1**) and (**B**) introduced in Chapter 1. We also keep the assumption (**H2.1**) on the operator A.

3.2 Local existence and global continuation of integral solutions

Throughout this section, we consider the autonomous case of Equation (2.1):

$$\begin{cases} \dfrac{d}{dt}x(t) = Ax(t) + F(x_t), \ t \geq 0, \\ x_0 = \varphi \in \mathcal{B}. \end{cases} \tag{3.2}$$

Contrarily to the results obtained in Chapter 2, to prove the local existence of solutions to Equation (3.2), we suppose a local Lipschitz condition on the nonlinear part F, that is,
(**H3.1**) F is Lipschitz continuous on the seminorm-balls of $\mathcal{B}$, i.e., for each $r > 0$ there exists a constant $c_0(r) > 0$ such that if $\varphi_1, \varphi_2 \in \mathcal{B}$ and $\|\varphi_1\|_{\mathcal{B}}, \|\varphi_2\|_{\mathcal{B}} \leq r$ then,

$$|F(\varphi_1) - F(\varphi_2)| \leq c_0(r) \|\varphi_1 - \varphi_2\|_{\mathcal{B}}.$$

Theorem 3.2.1 *Assume that* (**H2.1**) *and* (**H3.1**) *hold. Let $\varphi \in \mathcal{B}$ such that $\varphi(0) \in \overline{D(A)}$. Then, there exists a maximal interval of existence $(-\infty, b_\varphi)$, $b_\varphi > 0$, and a unique integral solution $x(., \varphi)$ of Equation (3.2), defined on $(-\infty, b_\varphi)$ and either*

$$b_\varphi = +\infty \quad or \quad \limsup_{t \to b_\varphi^-} |x(t, \varphi)| = +\infty.$$

Moreover, $x(t, \varphi)$ is a continuous function of φ, in the sense that if $\varphi \in \mathcal{B}$, $\varphi(0) \in \overline{D(A)}$ and $t \in [0, b_\varphi)$, then there exist positive constants α and ε such that, for $\psi \in \mathcal{B}$, $\psi(0) \in \overline{D(A)}$ and $\|\varphi - \psi\|_{\mathcal{B}} < \varepsilon$, we have $t \in [0, b_\varphi)$ and

$$|x(s, \varphi) - x(s, \psi)| \leq \alpha \|\varphi - \psi\|_{\mathcal{B}}, \quad for\ all\ s \in [0, t].$$

Proof. Note that (**H3.1**) implies that, for each $r > 0$ there exists $c_0(r) > 0$ such that for $\varphi \in \mathcal{B}$ and $\|\varphi\|_{\mathcal{B}} \leq r$, we have

$$|F(\varphi)| \leq c_0(r) \|\varphi\|_{\mathcal{B}} + |F(0)| \leq c_0(r) r + |F(0)|.$$

Let $\varphi \in \mathcal{B}$, $\varphi(0) \in \overline{D(A)}$, $r = \|\varphi\|_{\mathcal{B}} + 1$ and $c_1 = c_0(r)r + |F(0)|$. Define the function

$$y(t) = \begin{cases} S'(t)\varphi(0), & \text{for } t \geq 0, \\ \varphi(t), & \text{for } t \leq 0. \end{cases}$$

By virtue of the axioms $(\mathbf{A}-\mathbf{i})$ and $(\mathbf{A1})$, we deduce that $y_t \in \mathcal{B}$ and the map $t \to y_t$ is continuous. Then, for $b_1 \in (0,1)$, there exists $b_2 \in (0,1)$ such that

$$\|y_t - \varphi\|_{\mathcal{B}} \leq b_1, \text{ for all } t \in [0, b_2].$$

Without loss of generality, we can assume that $\omega > 0$. Let $0 < b \leq b_2$ such that

$$K_b \bar{M} e^{\omega b} c_1 b < 1 - b_1.$$

Let us introduce again the space

$$\mathbb{F}_b := \{u : (-\infty, b] \to E, \text{ such that } u_0 \in \mathcal{B} \text{ and } u : [0,b] \to E \text{ is continuous}\},$$

endowed with the seminorm $\|.\|_{\mathbb{F}_b}$, defined by

$$\|u\|_{\mathbb{F}_b} = \|u_0\|_{\mathcal{B}} + \sup_{0 \leq s \leq b} |u(s)|.$$

We can see from Axiom $(\mathbf{B})$, that $(\mathbb{F}_b, \|.\|_{\mathbb{F}_b})$ is complete.
Consider the following set

$$\mathbb{F}_b^{\varphi} := \left\{ u \in \mathbb{F}_b : \|u_0 - \varphi\|_{\mathcal{B}} = 0 \text{ and } \sup_{0 \leq t \leq b} \|u_t - \varphi\|_{\mathcal{B}} \leq 1 \right\}.$$

We can see as in the proof of Theorem 2.2.3 that $\mathbb{F}_b^{\varphi}$ is a nonempty closed subset of $\mathbb{F}_b$.
Consider now the mapping $\mathcal{T}$ defined on $\mathbb{F}_b^{\varphi}$ by

$$\begin{cases} (\mathcal{T}u)(t) = S'(t)\varphi(0) + \dfrac{d}{dt}\displaystyle\int_0^t S(t-s)F(u_s)ds, & \text{for } t \in [0,b], \\ (\mathcal{T}u)_0 = \varphi. \end{cases} \tag{3.3}$$

$\mathcal{T}$ maps elements of $\mathbb{F}_b^{\varphi}$ into $\mathbb{F}_b^{\varphi}$. In fact, by $(\mathbf{H3.1})$ and Axiom $(\mathbf{A})$ for every $u \in \mathbb{F}_b^{\varphi}$ the function $s \mapsto F(u_s)$ is continuous on $[0,b]$. Then, Proposition 1.2.9 implies that $s \mapsto \int_0^t S(t-s)F(u_s)ds$ is continuously differentiable on $[0,b]$. From this, $v := \mathcal{T}u$ is continuous on $[0,b]$ and $v \in \mathbb{F}_b$. To prove that $v \in \mathbb{F}_b^{\varphi}$, taking $w = v - y$. We get, for any $t \in [0,b]$,

$$\|v_t - \varphi\|_{\mathcal{B}} \leq \|w_t\|_{\mathcal{B}} + \|y_t - \varphi\|_{\mathcal{B}} \leq \|w_t\|_{\mathcal{B}} + b_1.$$

Since $\|u_s - \varphi\|_{\mathcal{B}} \leq 1$, for $s \in [0, b]$ and $r = \|\varphi\|_{\mathcal{B}} + 1$, we deduce that $\|u_s\|_{\mathcal{B}} \leq r$, for $s \in [0, b]$. Then,

$$|F(u_s)| \leq c_0(r) \|u_s\|_{\mathcal{B}} + |F(0)| \leq c_1.$$

On the other hand, by Axiom $(\mathbf{A} - \mathbf{iii})$ we see that for any $t \in [0, b]$

$$\|w_t\|_{\mathcal{B}} \leq K_b \sup_{0 \leq s \leq t} |w(s)|,$$

and for suitable constants $\bar{M}$ and ω

$$\begin{aligned} |w(t)| &= \left| \frac{d}{dt} \int_0^t S(t-s) F(u_s) ds \right|, \\ &\leq \bar{M} e^{\omega t} c_1 \int_0^t e^{-\omega s} ds, \\ &\leq \bar{M} e^{\omega t} c_1 b, \\ &< \frac{1}{K_b}(1 - b_1). \end{aligned}$$

Then, we deduce that

$$\|v_t - \varphi\|_{\mathcal{B}} < 1, \quad \text{for any } t \in [0, b].$$

Which implies that $v \in \mathbb{F}_b^{\varphi}$.

To prove that $\mathcal{T}$ is a strict contraction in $\mathbb{F}_b^{\varphi}$, we consider $u, v \in \mathbb{F}_b^{\varphi}$ and $t \in [0, b]$. Since

$$\|(\mathcal{T}u)_0 - (\mathcal{T}v)_0\|_{\mathcal{B}} = 0,$$

we obtain

$$\|\mathcal{T}u - \mathcal{T}v\|_{\mathbb{F}_b} = \sup_{t \in [0,b]} |(\mathcal{T}u)(t) - (\mathcal{T}v)(t)|,$$

and

$$\begin{aligned} |(\mathcal{T}u)(t) - (\mathcal{T}v)(t)| &= \left| \frac{d}{dt} \int_0^t S(t-s)(F(u_s) - F(v_s)) ds \right|, \\ &\leq \bar{M} e^{\omega t} \int_0^t |F(u_s) - F(v_s)| \, ds, \\ &\leq \bar{M} e^{\omega t} c_0(r) \int_0^t \|u_s - v_s\|_{\mathcal{B}} \, ds, \\ &\leq \bar{M} e^{\omega b} c_0(r) K_b b \|u - v\|_{\mathbb{F}_b}. \end{aligned}$$

Then,

$$\|\mathcal{T}u - \mathcal{T}v\|_{\mathbb{F}_b} \leq \bar{M} e^{\omega b} c_0(r) K_b b \|u - v\|_{\mathbb{F}_b}.$$

Note that $r \geq 1$, then

$$\bar{M}e^{\omega b}c_0(r)K_b b \quad \leq \bar{M}e^{\omega b}c_1 K_b b < 1.$$

This means that $\mathcal{T}$ is a strict contraction in $\left(\mathbb{F}_b^\varphi, \|.\|_{\mathbb{F}_b}\right)$. So, $\mathcal{T}$ has a fixed point x in $\mathbb{F}_b^\varphi$. Let u be another fixed point of $\mathcal{T}$. Then,

$$\begin{cases} \|u - x\|_{\mathbb{F}_b} = 0, \\ u_0 = x_0 = \varphi. \end{cases}$$

From the expression of the seminorm $\|.\|_{\mathbb{F}_b}$, we obtain

$$u(t) = x(t), \quad \text{for } t \in [0, b].$$

This means that $\mathcal{T}$ has one and only one fixed point x in $\mathbb{F}_b^\varphi$. We conclude that Equation (3.2) has one and only one integral solution which is defined on the interval $(-\infty, b]$.

Let $[0, b_\varphi)$, $b_\varphi > 0$, be the maximal interval of existence of the integral solution $x(., \varphi)$ of Equation (3.2). Assume that $b_\varphi < +\infty$ and $\limsup_{t \to b_\varphi^-} |x(t, \varphi)| < +\infty$. Then, from Axiom $(\mathbf{A} - \mathbf{iii})$, we can see that there exists a constant $r > 0$ such that $\|x_s(., \varphi)\|_{\mathcal{B}} \leq r$, for all $s \in [0, b_\varphi)$. Consider $t, t+h \in [0, b_\varphi)$ and $h > 0$,

$$\begin{aligned}
|x(t+h) - x(t)| &\leq |S'(t+h)\varphi(0) - S'(t)\varphi(0)| \\
&\quad + \left| \frac{d}{dt} \int_0^{t+h} S(t+h-s)F(x_s(., \varphi))ds - \frac{d}{dt} \int_0^t S(t-s)F(x_s(., \varphi))ds \right|, \\
&\leq |S'(t)\left(S'(h) - I\right)\varphi(0)| + \left| \frac{d}{dt} \int_t^{t+h} S(t+h-s)F(x_s(., \varphi))ds \right| \\
&\quad + \left| \frac{d}{dt} \int_0^t S(s)\left(F(x_{t+h-s}(., \varphi)) - F(x_{t-s}(., \varphi))\right) ds \right|, \\
&\leq \bar{M}e^{\omega b_\varphi} \left|\left(S'(h) - I\right)\varphi(0)\right| + \bar{M}e^{\omega b_\varphi} c_1 h \\
&\quad + \bar{M}e^{\omega b_\varphi} c_0(r) \int_0^t \|x_{s+h}(., \varphi) - x_s(., \varphi)\|_{\mathcal{B}}\, ds, \\
&\leq \bar{M}e^{\omega b_\varphi} |S'(h)\varphi(0) - \varphi(0)| + \bar{M}e^{\omega b_\varphi} c_1 h + b_\varphi M_{b_\varphi} \|x_h(., \varphi) - \varphi\|_{\mathcal{B}} \\
&\quad + \bar{M}e^{\omega b_\varphi} c_0(r) K_{b_\varphi} \int_0^t \sup_{0 \leq \sigma \leq s} |x(\sigma + h, \varphi) - x(\sigma, \varphi)|\, ds.
\end{aligned}$$

Using the same reasoning, one can show a similar result for $h < 0$. By applying the well-known Gronwall lemma, we can deduce that x is uniformly continuous on $[0, b_\varphi)$. Consequently, $x(., \varphi)$ can be extended to b_φ, which contradicts the maximality of $[0, b_\varphi)$.

We prove now that the solution depends continuously on the initial data. Let $\varphi \in \mathcal{B}$ such that $\varphi(0) \in \overline{D(A)}$ and $t \in [0, b_\varphi)$ be fixed. Set

$$r = 1 + \sup_{0 \le s \le t} \|x_s(., \varphi)\|_{\mathcal{B}}.$$

Let $\varepsilon \in (0,1)$ and

$$\psi \in \mathcal{B},\ \psi(0) \in \overline{D(A)} \quad \text{such that} \quad \|\varphi - \psi\|_{\mathcal{B}} < \varepsilon.$$

It is clear that

$$\|\psi\|_{\mathcal{B}} \le \|\varphi\|_{\mathcal{B}} + \varepsilon < r.$$

We put

$$\begin{cases} b_0 = & \sup\left\{s > 0 : \|x_\sigma(., \psi)\|_{\mathcal{B}} \le r, \text{ for all } \sigma \in [0, s]\right\}, \\ c(t) = & (M_{b_0} + HK_{b_0}\bar{M}e^{\omega t}) \exp\left(K_{b_0}\bar{M}e^{\omega t}c_0(r)t\right). \end{cases}$$

We choose ε small enough such that

$$c(t)\varepsilon < 1.$$

Suppose that $b_0 < t$. We obtain for $s \in [0, b_0]$,

$$\begin{aligned} \|x_s(., \varphi) - x_s(., \psi)\|_{\mathcal{B}} &\le M_{b_0} \|\varphi - \psi\|_{\mathcal{B}} + K_{b_0} \sup_{0 \le \xi \le s} |x(\xi, \varphi) - x(\xi, \psi)|, \\ &\le M_{b_0} \|\varphi - \psi\|_{\mathcal{B}} + K_{b_0} \sup_{0 \le \xi \le s} \Big\{ \left|S'(\xi)\left(\varphi(0) - \psi(0)\right)\right| \\ &\quad + \left| \frac{d}{d\xi} \int_0^\xi S(\xi - \sigma)\left(F(x_\sigma(., \varphi)) - F(x_\sigma(., \psi))\right) d\sigma \right| \Big\}, \\ &\le \left(HK_{b_0}\bar{M}e^{\omega t} + M_{b_0}\right) \|\varphi - \psi\|_{\mathcal{B}} \\ &\quad + K_{b_0}\bar{M}e^{\omega t}c_0(r) \int_0^s \|x_\sigma(., \varphi) - x_\sigma(., \psi)\|_{\mathcal{B}}\, d\sigma. \end{aligned}$$

By Gronwall's lemma, we deduce that

$$\|x_s(, \varphi) - x_s(, \psi)\|_{\mathcal{B}} \le c(t) \|\varphi - \psi\|_{\mathcal{B}}. \tag{3.4}$$

This implies that

$$\|x_s(., \psi)\|_{\mathcal{B}} \le c(t)\varepsilon + r - 1 < r, \text{ for all } s \in [0, b_0].$$

By continuity, there exists $\delta > 0$ such that

$$\|x_s(., \psi)\|_{\mathcal{B}} \le c(t)\varepsilon + r - 1 < r, \quad \text{for all } s \in [0, b_0 + \delta].$$

It follows that b_0 cannot be the largest number $s > 0$ such that $\|x_\sigma(.,\psi)\|_{\mathcal{B}} \leq r$, for $\sigma \in [0,s]$. Thus, $b_0 \geq t$ and $t < b_\psi$. Furthermore, $\|x_s(.,\psi)\|_{\mathcal{B}} \leq r$, for $s \in [0,t]$. Then, using the inequality (3.4), we deduce the continuous dependence on the initial data. This completes the proof. ■

3.3 Existence of strict solutions

As in the previous chapter, we can obtain the strictness of the integral solution under similar restrictive conditions: (**H3.1**) and

(**H3.2**) $F : \mathcal{B} \to E$ is continuously differentiable and $F' : \mathcal{B} \to \mathcal{L}(\mathcal{B}, E)$ is locally Lipschitz continuous (in the sense of (**H3.1**)).

Theorem 3.3.1 *Assume that the phase space $\mathcal{B}$ is a normed space which satifies Axiom* (**C1**) *or Axiom* (**D1**) *and the conditions* (**H2.1**), (**H3.1**) *and* (**H3.2**) *hold. Let $\varphi \in \mathcal{B}$ be a continuously differentiable function with $\varphi' \in \mathcal{B}$ and*

$$\varphi(0) \in D(A), \;\; \varphi'(0) \in \overline{D(A)} \;\; \text{and } \varphi'(0) = A\varphi(0) + F(\varphi).$$

If $x(.,\varphi) : (-\infty, b_\varphi) \to E$ is the integral solution of Equation (3.2) given by Theorem 3.2.1, then $x(.,\varphi)$ is a strict solution of Equation (3.2).

Proof. Let $\varphi \in \mathcal{B}$ such that $\varphi(0) \in D(A)$, $\varphi'(0) \in \overline{D(A)}$ and $\varphi'(0) = A\varphi(0) + F(\varphi)$. Let $x := x(.,\varphi)$ be the unique integral solution of Equation (3.2) on $(-\infty, b_\varphi)$ and $b_1 \in (0, b_\varphi)$. Consider the following equation

$$\begin{cases} \dfrac{dy}{dt}(t) = Ay(t) + F'(x_t)y_t, \;\; t \geq 0, \\ y_0 = \varphi' \in \mathcal{B}. \end{cases}$$

Using a similar argument as in the prove of Theorem 3.2.1, we can show that the above equation has a unique integral solution $y : [0, b_1] \to E$, which is given by

$$y(t) = \begin{cases} S'(t)\varphi'(0) + \dfrac{d}{dt}\displaystyle\int_0^t S(t-s)F'(x_s)y_s ds, & \text{for } t \in [0, b_1], \\ \varphi'(t), & \text{for } t \in (-\infty, 0]. \end{cases}$$

Define the function z by

$$z(t) = \begin{cases} \varphi(0) + \displaystyle\int_0^t y(s)ds, & \text{for } t \in [0, b_1], \\ \varphi(t), & \text{for } t \in (-\infty, 0]. \end{cases} \tag{3.5}$$

We will prove that $x = z$. Using the expression of y, we obtain

$$z(t) = \varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)F'(x_s)y_s ds, \quad \text{for } t \in [0, b_1].$$

From $\varphi(0) \in D(A)$, $\varphi'(0) \in \overline{D(A)}$ and $\varphi'(0) = A\varphi(0) + F(\varphi)$, we deduce

$$S(t)\varphi'(0) = S(t)A\varphi(0) + S(t)F(\varphi).$$

Using Corollary 1.2.2, we get

$$S(t)\varphi'(0) = S'(t)\varphi(0) - \varphi(0) + S(t)F(\varphi).$$

On the other hand, by use of Lemma 2.2.6 or Lemma 2.2.7, we can infer from (3.5) that

$$z_t = \varphi + \int_0^t y_s ds, \quad \text{for } t \in [0, b_1].$$

Consequently, the maps $t \mapsto z_t$ and $t \mapsto \int_0^t S(t-s)F(z_s)\,ds$ are continuously differentiable and we have

$$\frac{d}{dt}\int_0^t S(t-s)F(z_s)\,ds = S(t)F(\varphi) + \int_0^t S(t-s)F'(z_s)\,y_s ds.$$

So, we deduce that

$$S(t)F(\varphi) = \frac{d}{dt}\int_0^t S(t-s)F(z_s)\,ds - \int_0^t S(t-s)F'(z_s)\,y_s ds.$$

Consequently, z satisfies for $t \in [0, b_1]$

$$z(t) = S'(t)\varphi(0) + S(t)F(\varphi) + \int_0^t S(t-s)F'(z_s)y_s ds.$$

This implies that

$$\begin{aligned} z(t) = \; & S'(t)\varphi(0) + \frac{d}{dt}\int_0^t S(t-s)F(z_s)\,ds \\ & - \int_0^t S(t-s)F'(z_s)\,y_s ds + \int_0^t S(t-s)F'(x_s)\,y_s ds. \end{aligned}$$

Therefore,

$$\begin{aligned} x(t) - z(t) = & \frac{d}{dt}\int_0^t S(t-s)\left(F(x_s) - F(z_s)\right) ds \\ & - \int_0^t S(t-s)\left(F'(x_s) - F'(z_s)\right) y_s ds. \end{aligned}$$

Then, we deduce

$$\begin{aligned} |x(t) - z(t)| \leq \beta e^{\omega b_1} & \left(\int_0^t |F(x_s) - F(z_s)|\, ds \right. \\ & \left. + \int_0^t \|F'(x_s) - F'(z_s)\|_{\mathcal{L}(\mathcal{B},E)} \|y_s\|_{\mathcal{B}}\, ds \right). \end{aligned}$$

Let

$$r = \max\left(\sup_{0\leq s\leq b_1} \|x_s\|_{\mathcal{B}},\ \sup_{0\leq s\leq b_1} \|y_s\|_{\mathcal{B}},\ \sup_{0\leq s\leq b_1} \|z_s\|_{\mathcal{B}} \right).$$

There exist $c_0(r)$, $v_0(r) \geq 0$ such that if $\varphi_1, \varphi_2 \in \mathcal{B}$ and $\|\varphi_1\|_{\mathcal{B}}, \|\varphi_2\|_{\mathcal{B}} \leq r$, then

$$\begin{cases} |F(\varphi_1) - F(\varphi_2)| \leq c_0(r)\, \|\varphi_1 - \varphi_2\|_{\mathcal{B}}, \\ \|F'(\varphi_1) - F'(\varphi_2)\|_{\mathcal{L}(\mathcal{B},E)} \leq v_0(r)\, \|\varphi_1 - \varphi_2\|_{\mathcal{B}}. \end{cases}$$

This implies that

$$\sup_{0\leq s\leq t} |x(s) - z(s)| \leq \beta e^{\omega b_1}\left(c_0(r) + r v_0(r)\right) \int_0^t \|x_s - z_s\|_{\mathcal{B}}\, ds.$$

Applying Axiom ($\mathbf{A1} - \mathbf{iii}$), we obtain

$$\|x_t - z_t\|_{\mathcal{B}} \leq K_{b_1} \sup_{0\leq s\leq t} |x(s) - z(s)|.$$

Thus, we obtain

$$\|x_t - z_t\|_{\mathcal{B}} \leq K_{b_1}\beta e^{\omega b_1}\left(c_0(r) + r\, v_0(r)\right) \int_0^t \|x_s - z_s\|_{\mathcal{B}}\, ds.$$

By Gronwall's lemma, $\|x_t - z_t\|_{\mathcal{B}} = 0$ for any $t \in [0, b_1]$. Using Axiom ($\mathbf{A1} - \mathbf{ii}$), we conclude that $x(t) = z(t)$, for all $t \in (-\infty, b_1]$. Consequently, the integral solution x is continuously differentiable. Then, from Proposition 2.2.2 we conclude that x is a strict solution of Equation (3.2). This completes the proof of the theorem. ■

3.4 The solution semigroup and linearized stability

In this section, we give some properties of the solution map associated to Equation (3.2). We make the following autonomous version of (**H2.4**).

(**H3.3**) F is globally Lipschitz continuous on $\mathcal{B}$:

$$|F(\varphi_1) - F(\varphi_2)| \leq L \|\varphi_1 - \varphi_2\|_{\mathcal{B}}, \quad \text{for } \varphi_1, \varphi_2 \in \mathcal{B}.$$

Using this assumption and a fixed point theorem, we have shown in Theorem 2.2.5 that for all $\varphi \in \mathcal{B}$ such that $\varphi(0) \in \overline{D(A)}$, Equation (3.2) has a unique integral solution which is defined on $\mathbb{R}$.

Let

$$\mathcal{X} := \left\{ \varphi \in \mathcal{B} : \varphi(0) \in \overline{D(A)} \right\}.$$

Define the following operator $\mathcal{U}(t)$ on $\mathcal{X}$, for $t \geq 0$ by

$$\mathcal{U}(t)\varphi = x_t(., \varphi),$$

where $x(., \varphi)$ is the unique integral solution of Equation (3.2). We can prove the following result.

Proposition 3.4.1 *The family* $(\mathcal{U}(t))_{t\geq 0}$ *is a strongly continuous semigroup on* $\mathcal{X}$*, that is*

(**i**) $\mathcal{U}(0) = I$*,*

(**ii**) $\mathcal{U}(t + s) = \mathcal{U}(t)\mathcal{U}(s)$*, for all* $t, s \geq 0$*,*

(**iii**) *for all* $\varphi \in \mathcal{X}$*,* $\mathcal{U}(t)\varphi$ *is a continuous function of* $t \geq 0$ *with values in* $\mathcal{X}$*.*

Moreover,

(**iv**) $(\mathcal{U}(t))_{t\geq 0}$ *satisfies, for* $t \geq 0$ *and* $\theta \in (-\infty, 0]$*, the following translation property*

$$(\mathcal{U}(t)\varphi)(\theta) = \begin{cases} (\mathcal{U}(t+\theta)\varphi)(0), & \text{if } t + \theta \geq 0, \\ \varphi(t+\theta), & \text{if } t + \theta \leq 0, \end{cases}$$

(**v**) *there exist two positive locally bounded functions* $m(.), n(.) : \mathbb{R}^+ \to \mathbb{R}^+$ *such that, for all* $\varphi_1, \varphi_2 \in \mathcal{X}$ *and* $t \geq 0$*,*

$$\|\mathcal{U}(t)\varphi_1 - \mathcal{U}(t)\varphi_2\|_{\mathcal{B}} \leq m(t)e^{n(t)} \|\varphi_1 - \varphi_2\|_{\mathcal{B}}.$$

Proof. (**i**) and (**ii**) are consequence of the uniqueness of the integral solution of Equation (3.2).

(**iii**) and (**iv**) can be proved by using the integral Equation (2.5) and some properties of $\mathcal{B}$. To prove (**v**), we can see that

$$\begin{aligned}
&\|\mathcal{U}(t)\phi - \mathcal{U}(t)\psi\|_{\mathcal{B}} = \|x_t(.,\phi) - x_t(.,\psi)\|_{\mathcal{B}}\,,\\
&\leq K(t) \sup_{0\leq s\leq t} |x(s,\phi) - x(s,\psi)| + M(t)\,\|\phi-\psi\|_{\mathcal{B}}\,,\\
&\leq K(t) \sup_{0\leq s\leq t} |S'(s)(\phi(0) - \psi(0))| + M(t)\,\|\phi-\psi\|_{\mathcal{B}}\,,\\
&+K(t) \sup_{0\leq s\leq t} \left|\frac{d}{ds}\int_0^s S(s-\sigma)\left(F(x_\sigma(.,\phi)) - F(x_\sigma(.,\psi))\right)d\sigma\right|.
\end{aligned}$$

From Proposition 1.2.9, we deduce that

$$\begin{aligned}
\|\mathcal{U}(t)\phi - \mathcal{U}(t)\psi\|_{\mathcal{B}} \leq \; & K(t)\bar{M}e^{\omega t}H\,\|\phi-\psi\|_{\mathcal{B}} + M(t)\,\|\phi-\psi\|_{\mathcal{B}}\\
& +2K(t)l(t)L\int_0^t \|x_\sigma(.,\phi) - x_\sigma(.,\psi)\|_{\mathcal{B}}\,d\sigma.
\end{aligned}$$

This completes the proof by using Gronwall's lemma. ■

We will now give another property of $(\mathcal{U}(t))_{t\geq 0}$. We need to make the compactness assumption (**H2.2**).

Proposition 3.4.2 *Assume that* (**H2.1**), (**H2.2**), *and* (**H3.3**) *hold. Then, the semigroup* $(\mathcal{U}(t))_{t\geq 0}$ *is decomposed on* $\mathcal{X}$ *as follows*

$$\mathcal{U}(t) = \mathcal{U}_1(t) + \mathcal{U}_2(t), \quad t\geq 0,$$

where $(\mathcal{U}_2(t))_{t\geq 0}$ *is the semigroup on* $\mathcal{X}$ *given, for* $\varphi \in \mathcal{X}$, $t\geq 0$ *and* $\theta \in (-\infty, 0]$, *by*

$$\left(\mathcal{U}_2(t)\varphi\right)(\theta) = \begin{cases} S'(t+\theta)\varphi(0), & t+\theta \geq 0,\\ \varphi(t+\theta), & t+\theta\leq 0,\end{cases}$$

and $\mathcal{U}_1(t)$ *is compact on* $\mathcal{X}$ *for* $t>0$.

Proof. Let $(\psi_n)_{n\geq 0}$ be a bounded sequence in $\mathcal{X}$. First, we prove that the sequence $((\mathcal{U}_1(t)\psi_n)(\theta))_{n\geq 0}\,, t>0, \theta\in(-\infty,0]$, is totally bounded in E and the sequence $(\mathcal{U}_1(t)\psi_n)_{n\geq 0}$, $t>0$, is equicontinuous in $(-\infty, 0]$. Let $t>0$ and $\theta\in(-\infty,0]$ be fixed. We have

$$(\mathcal{U}_1(t)\psi_n)(\theta) = \begin{cases} \left(\mathcal{U}_1(t+\theta)\psi_n\right)(0), & t+\theta\geq 0,\\ 0, & t+\theta \leq 0.\end{cases}$$

Let $0<\varepsilon<t+\theta$. Then, for $\lambda>\omega$, we have

$$\lambda R(\lambda, A)(\mathcal{U}_1(t)\psi_n)(\theta) \;=\; \int_0^{t+\theta} S'(t+\theta-s)\lambda R(\lambda,A)F(x_s(.,\psi_n))ds,$$

$$= S'(\varepsilon)\int_0^{t+\theta-\varepsilon} S'(t+\theta-\varepsilon-s)\lambda R(\lambda,A)F(x_s(.,\psi_n))ds,$$

$$+\int_{t+\theta-\varepsilon}^{t+\theta} S'(t+\theta-s)\lambda R(\lambda,A)F(x_s(.,\psi_n))ds,$$

where $x(.,\psi_n)$ is the unique integral solution of Equation (3.2) for the initial condition ψ_n. Since $S'(\varepsilon)$ is compact, there exists a compact set W_ε such that

$$S'(\varepsilon)\left\{\int_0^{t+\theta-\varepsilon} S'(t+\theta-\varepsilon-s)\lambda R(\lambda,A)F(x_s(.,\psi_n))ds : n\geq 0,\ \lambda>\omega\right\}\subset W_\varepsilon\,.$$

Furthermore, for all $n\geq 0$

$$\left|\int_{t+\theta-\varepsilon}^{t+\theta} S'(t+\theta-s)\lambda R(\lambda,A)F(x_s(.,\psi_n))ds\right| \leq \frac{\lambda\bar{M}}{\lambda-\omega}\overline{M}_t\alpha_t\varepsilon,$$

where

$$\alpha_t=\sup_{s\in[0,t]}\{|F(x_s(.,\psi_n)|:n\geq 0\} \quad\text{and}\quad \overline{M}_t=\sup_{s\in[0,t]}\left\|S'(s)\right\|_{\overline{D(A)}}.$$

This shows, by letting λ to $+\infty$, that $((\mathcal{U}_1(t)\psi_n)(\theta))_{n\geq 0}$, $t>0$, $\theta\in(-\infty,0]$, is totally bounded. To establish the equicontinuity, let $t>0$ and $\theta_0\in(-\infty,0]$ be fixed. For $\theta\in(-\infty,0]$ such that $|\theta-\theta_0|$ small enough and $\theta_0<\theta$, we obtain

$$(\mathcal{U}_1(t)\psi_n)(\theta)-(\mathcal{U}_1(t)\psi_n)(\theta_0)=$$

$$\begin{cases} (\mathcal{U}_1(t+\theta)\psi_n)(0)-(\mathcal{U}_1(t+\theta_0)\psi_n)(0), & t+\theta_0>0,\\ (\mathcal{U}_1(\theta-\theta_0)\psi_n)(0), & t+\theta_0=0,\\ 0, & t+\theta_0<0.\end{cases}$$

Therefore, for $-t<\theta_0<\theta\leq 0$ and $\lambda>\omega$, we obtain

$$\begin{aligned}&\lambda R(\lambda,A)\left((\mathcal{U}_1(t)\psi_n)(\theta)-(\mathcal{U}_1(t)\psi_n)(\theta_0)\right)\\ &\quad=\int_0^{t+\theta_0}\left(S'(t+\theta-s)-S'(t+\theta_0-s)\right)\lambda R(\lambda,A)F(x_s(.,\psi_n))ds\\ &\quad+\int_{t+\theta_0}^{t+\theta} S'(t+\theta-s)\lambda R(\lambda,A)F(x_s(.,\psi_n))ds.\end{aligned}$$

Which leads to

$$\begin{aligned}
&|\lambda R(\lambda, A)\left((\mathcal{U}_1(t)\psi_n)(\theta) - (\mathcal{U}_1(t)\psi_n)(\theta_0)\right)| \\
&\quad \leq \left| (S'(\theta-\theta_0) - I) \int_0^{t+\theta_0} S'(t+\theta_0 - s)\lambda R(\lambda, A)F(x_s(.,\psi_n))ds \right| \\
&\quad\quad + \frac{\lambda \bar{M}}{\lambda - \omega} \overline{M}_t \alpha_t(\theta - \theta_0).
\end{aligned}$$

Besides, there exists a compact set W such that

$$\left\{ \int_0^{t+\theta_0} \left| S'(t+\theta_0 - s)\lambda R(\lambda, A)F(x_s(.,\psi_n)) \right| ds : n \geq 0,\ \lambda > \omega \right\} \subseteq W.$$

Letting λ to $+\infty$ and using the fact that $(S'(.)x)_{x\in W}$ is equicontinuous at the right in 0, gives

$$\lim_{\substack{\theta \to \theta_0 \\ \theta > \theta_0}} |(\mathcal{U}_1(t)\psi_n)(\theta) - (\mathcal{U}_1(t)\psi_n)(\theta_0)| = 0.$$

By a similar argument we deduce, for $-\infty < \theta < \theta_0 \leq 0$, the claimed equicontinuity. Proceeding by Arzéla-Ascoli's theorem, there is a continuous function $\phi : (-\infty, 0] \to E$ and a subsequence ϕ_n of $(\mathcal{U}_2(t)\psi_n)_{n\geq 0}$ which converges compactly to ϕ in $(-\infty, 0]$. In addition ϕ is continuous and $\phi(\theta) = 0$ if $\theta \leq -t$, because $\phi_n(\theta) = 0$ for $\theta \leq -t$ and $n \geq 0$. Hence ϕ belongs to $\mathcal{B}$ by Axiom $(\mathbf{A}-\mathbf{i})$, and moreover we get $\|\phi_n - \phi\|_{\mathcal{B}} \to 0$ as $n \to \infty$ because of $\|\phi_n - \phi\|_{\mathcal{B}} \leq K(t) \sup_{-t\leq\theta\leq 0} |\phi_n(\theta) - \phi(\theta)|$ by Axiom $(\mathbf{A}-\mathbf{iii})$. In definitive, we have proved that the image of any bounded sequence contains a converging subsequence in $\mathcal{B}$ with respect to the seminorm. ■

We focus now our attention on the stability near an equilibrium of the nonlinear semigroup $(\mathcal{U}(t))_{t\geq 0}$ on $\mathcal{X}$. We keep the assumptions (**H2.1**) and (**H3.3**), and we add the following:

(**H3.4**) F is differentiable at 0 with respect to $(\mathcal{B}, \|.\|_{\mathcal{B}})$ and $F(0) = 0$.

Consider the linearized equation of (3.2) corresponding to the derivative $F'(0)$ at 0 :

$$\begin{cases} \dfrac{dx}{dt}(t) = Ax(t) + L(x_t),\ t \geq 0, \\ x_0 = \varphi \in \mathcal{B}, \end{cases} \tag{3.6}$$

where $L := F'(0)$ and $(\mathcal{T}(t))_{t\geq 0}$ is the solution semigroup on $\mathcal{X}$ associated to Equation (3.6).

Proposition 3.4.3 *Assume that* (**H2.1**), (**H3.3**) *and* (**H3.4**) *hold. Then, the derivative at zero of the nonlinear semigroup* $(\mathcal{U}(t))_{t\geq 0}$*, associated to Equation (3.2) is the linear semigroup* $(\mathcal{T}(t))_{t\geq 0}$ *associated to Equation (3.6).*

Proof. It suffices to show that for each $t \geq 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|\mathcal{U}(t)\varphi - \mathcal{T}(t)\varphi\|_{\mathcal{B}} \leq \varepsilon \|\varphi\|_{\mathcal{B}}, \qquad \text{for } \|\varphi\|_{\mathcal{B}} \leq \delta.$$

Note that

$$\|\mathcal{U}(t)\varphi - \mathcal{T}(t)\varphi\|_{\mathcal{B}} = \|x_t(.,\varphi) - y_t(.,\varphi)\|_{\mathcal{B}},$$

where $x(.,\varphi)$ (respectively, $y(.,\varphi)$) is the unique integral solution of Equation (3.2) (respectively, Equation (3.6)).

From Axiom $(\mathbf{A}-\mathbf{iii})$, we see that for all $t \geq 0$

$$\begin{aligned}
\|\mathcal{U}(t)\varphi - \mathcal{T}(t)\varphi\|_{\mathcal{B}} &\leq K(t) \sup_{0\leq\sigma\leq t} |x(\sigma,\varphi) - y(\sigma,\varphi)|, \\
&\leq K(t) \sup_{0\leq\sigma\leq t} \left| \frac{d}{d\sigma} \int_0^\sigma S(\sigma - s)\left(F(\mathcal{U}(s)\varphi) - L(\mathcal{T}(s)\varphi)\right) ds \right|, \\
&\leq K(t)\bar{M}e^{\omega t} \int_0^t e^{-\omega s} |F(\mathcal{U}(s)\varphi) - L(\mathcal{T}(s)\varphi)| \, ds, \\
&\leq K(t)\bar{M}e^{\omega t} \left(\int_0^t e^{-\omega s} |F(\mathcal{U}(s)\varphi) - L(\mathcal{U}(s)\varphi)| \, ds \right. \\
&\quad \left. + \int_0^t e^{-\omega s} |L(\mathcal{U}(s)\varphi) - L(\mathcal{T}(s)\varphi)| \, ds \right).
\end{aligned}$$

By virtue of the differentiability of F at 0 and from $(\mathbf{v})$ of Proposition 3.4.1, we deduce that for $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_0^t e^{-\omega s} |F(\mathcal{U}(s)\varphi) - L(\mathcal{U}(s)\varphi)| \, ds \leq \varepsilon \|\varphi\|_{\mathcal{B}}, \qquad \text{for } \|\varphi\|_{\mathcal{B}} \leq \delta.$$

On the other hand, we find that

$$\int_0^t e^{-\omega s} |L(\mathcal{U}(s)\varphi) - L(\mathcal{T}(s)\varphi)| \, ds \leq \|L\|_{\mathcal{L}(\mathcal{B},E)} \int_0^t e^{-\omega s} \|\mathcal{U}(s)\varphi - \mathcal{T}(s)\varphi\|_{\mathcal{B}} \, ds.$$

Consequently,

$$\|\mathcal{U}(t)\varphi - \mathcal{T}(t)\varphi\|_{\mathcal{B}} \leq K(t)\bar{M}e^{\omega t} \left(\varepsilon \|\varphi\|_{\mathcal{B}} + \|L\|_{\mathcal{L}(\mathcal{B},E)} \int_0^t e^{-\omega s} \|\mathcal{U}(s)\varphi - \mathcal{T}(s)\varphi\|_{\mathcal{B}} \, ds \right).$$

By Gronwall's lemma, we deduce that

$$\|\mathcal{U}(t)\varphi - \mathcal{T}(t)\varphi\|_{\mathcal{B}} \leq K_t\bar{M}\varepsilon \|\varphi\|_{\mathcal{B}} \exp\left((\|L\|_{\mathcal{L}(\mathcal{B},E)} K_t\bar{M} + \omega)t \right), \; t \geq 0.$$

Hence, we conclude that $\mathcal{U}(t)$ is differentiable at 0 and $(D_\varphi \mathcal{U}(t))(0) = \mathcal{T}(t)$, for each $t \geq 0$.

■

Definition 3.4.1 *(Desh and Schappacher [41]) Let* $(V(t))_{t\geq 0}$ *be a strongly continuous semigroup on a Banach space* $(Y, |.|_Y)$. *A point* $x_0 \in Y$ *is called an equilibrium of* $(V(t))_{t\geq 0}$ *if* $V(t)x_0 = x_0$ *for all* $t \geq 0$. *An equilibrium* $x_0 \in Y$ *is called locally exponentially stable if there exist* $\delta > 0$, $\mu > 0$ *and* $k \geq 1$ *such that*

$$|V(t)x - x_0|_Y \leq ke^{-\mu t} |x - x_0|_Y, \quad \text{for } t \geq 0 \text{ and } x \in Y \text{ with } |x - x_0|_Y \leq \delta.$$

Theorem 3.4.4 *[41] Let* $(V(t))_{t\geq 0}$ *be a nonlinear strongly continuous semigroup in a Banach space* Y. *Assume that* $x_0 \in Y$ *is an equilibrium of* $(V(t))_{t\geq 0}$ *such that* $V(t)$ *is Fréchet-differentiable at* x_0 *for each* $t \geq 0$, *with* $W(t)$ *the Fréchet-derivative at* x_0 *of* $V(t)$. *Then,* $(W(t))_{t\geq 0}$ *is a strongly continuous semigroup of bounded linear operators on* Y. *Moreover, if* $(W(t))_{t\geq 0}$ *is exponentially stable, then* x_0 *is a locally exponentially stable equilibrium of* $(V(t))_{t\geq 0}$.

By using the above theorem in the Banach space of equivalence classes $\mathcal{B} \,/\, \|.\|_{\mathcal{B}}$, we get our main result concerning the stability of an equilibrium of $(\mathcal{U}(t))_{t\geq 0}$.

Theorem 3.4.5 *Assume that* (**H2.1**), (**H3.3**) *and* (**H3.4**) *hold. If* $(\mathcal{T}(t))_{t\geq 0}$ *is exponentially stable on* $\mathcal{X}$, *i.e., there exist constants* $\bar{M} \geq 1$ *and* $\overline{\omega} > 0$ *such that* $\|\mathcal{T}(t)\| \leq \bar{M}e^{-\overline{\omega}t}$, *for* $t \geq 0$, *then zero is a locally exponentially stable equilibrium of* $(\mathcal{U}(t))_{t\geq 0}$ *on* $\mathcal{X}$; *that is, there exist* $\delta > 0$, $\mu > 0$ *and* $k \geq 1$ *such that*

$$\|\mathcal{U}(t)\psi\|_{\mathcal{B}} \leq ke^{-\mu t} \|\psi\|_{\mathcal{B}}, \quad \text{for } t \geq 0 \text{ and } \psi \in \mathcal{X} \text{ with } \|\psi\|_{\mathcal{B}} \leq \delta.$$

To study the stability of solutions to the linearized equation (3.6), we need to characterize the infinitesimal generator of the linear semigroup $(\mathcal{T}(t))_{t\geq 0}$ on $\mathcal{X}$. To do this, we assume that $\mathcal{B}$ satisfies the additional axiom (**D1**) used in strictness of solutions.

Theorem 3.4.6 *Assume that* $\mathcal{B}$ *is a subspace of* $C((-\infty, 0]; E)$ *(the space of continuous functions from* $(-\infty, 0]$ *into* E*) satisfying Axioms* (**A**), (**A1**), (**B**) *and* (**D1**). *Then, the infinitesimal generator* $A_{\mathcal{T}}$ *of* $(\mathcal{T}(t))_{t\geq 0}$ *is given by*

$$\begin{cases} D(A_{\mathcal{T}}) = \{\varphi \in \mathcal{X} : \varphi' \in \mathcal{X}, \, \varphi(0) \in D(A) \text{ and } \varphi'(0) = A\varphi(0) + L(\varphi)\}, \\ A_{\mathcal{T}}\varphi = \varphi'. \end{cases}$$

Proof. Let $\varphi \in \mathcal{X}$ be continuously differentiable such that

$$\varphi' \in \mathcal{X}, \quad \varphi(0) \in D(A) \text{ and } \varphi'(0) = A\varphi(0) + L(\varphi).$$

Let $x(.,\varphi)$ be the integral solution of Equation (3.2). In the proof of Theorems 2.2.8 and 3.3.1, we showed that there exists a continuous mapping $y:\mathbb{R}\to E$ with $y_0=\varphi'$ such that

$$x_t(.,\varphi)=\varphi+\int_0^t y_s ds, \qquad \text{for } t\geq 0.$$

Which implies that $\lim\limits_{t\to 0^+}\frac{1}{t}\left(x_t(.,\varphi)-\varphi\right)$ exists and is equal to φ'. Then, $\varphi\in D(A_{\mathcal{T}})$. Conversely, let $\varphi\in\mathcal{X}$ such that

$$\lim_{t\to 0^+}\frac{1}{t}\left(x_t(.,\varphi)-\varphi\right)=\psi=A_{\mathcal{T}}\varphi, \quad \text{exists in } \mathcal{X}.$$

Axiom (**D1**) implies that

$$\lim_{t\to 0^+}\frac{1}{t}\left(x_t(\theta,\varphi)-\varphi(\theta)\right) \quad \text{exists for all } \theta\leq 0 \text{ and is equal to } \psi(\theta).$$

Then, for $\theta\in(-\infty,0)$, we have

$$\psi(\theta)=\lim_{t\to 0^+}\frac{1}{t}\left(\varphi(t+\theta)-\varphi(\theta)\right)=D^+\varphi(\theta),$$

where $D^+\varphi$ is the right derivative of φ, and $D^+\varphi=\psi$ in $(-\infty,0)$. Since ψ is continuous, $D^+\varphi$ is also continuous in $(-\infty,0)$. Let us introduce the following result.

Lemma 3.4.7 *[97] Let φ be continuous and differentiable on the right on $[a,b)$. If $D^+\varphi$ is continuous on $[a,b)$, then φ is continuously differentiable on $[a,b)$.*

From the above lemma, we deduce that the function φ is continuously differentiable in $(-\infty,0)$ and $\varphi'=\psi$. On the other hand, for $\theta=0$, one has $\lim\limits_{\theta\to 0^-}\varphi'(\theta)$ exists and equals to $\psi(0)$. From this, we infer that the function φ is continuously differentiable in $(-\infty,0]$ and $\varphi'=\psi\in\mathcal{X}$. Since $t\to x_t(.,\varphi)$ is continuously differentiable in $[0,+\infty)$ and using Axiom (**D1**), we can state that the map $t\to x(t,\varphi)$ is continuously differentiable in $[0,+\infty)$. Then, from Proposition 2.2.2, x is a strict solution of Equation (3.6), and for $\theta=0$, one has

$$\varphi'(0)=\psi(0)=\lim_{t\to 0}\frac{1}{t}\left(x(t)-\varphi(0)\right)=x'(0) \text{ and } x'(0)=A\varphi(0)+L(\varphi).$$

Form this we conclude that

$$\varphi'(0)=A\varphi(0)+L(\varphi).$$

This completes the proof of the theorem. ■

In the sequel, we give a necessary and sufficient condition to ensure the stability of the linear semigroup $(\mathcal{T}(t))_{t\geq 0}$ on $\mathcal{X}$. We need the following results.
Let $\mathcal{H}: Z \to Z$ be a closed linear operator with a dense domain $D(\mathcal{H})$ in a Banach space Z. We denote by $\sigma(\mathcal{H})$ the spectrum of $\mathcal{H}$, and by $\sigma_p(\mathcal{H})$ the point spectrum of $\mathcal{H}$.
We call essential spectrum of $\mathcal{H}$ (denotation by $\sigma_e(\mathcal{H})$) the set of all λ in $\sigma(\mathcal{H})$ for which at least one of the following holds:

(**i**) $\mathrm{Im}(\lambda I - \mathcal{H}) = \{(\lambda I - \mathcal{H})z : z \in Z\}$ is not closed,

(**ii**) the generalized eigenspace $M_\lambda(\mathcal{H}) = \underset{k\geq 1}{\cup} \ker(\lambda I - \mathcal{H})^k$ of λ is infinite dimensional,

(**iii**) λ is a limit point of $\sigma(\mathcal{H})$, that is, $\lambda \in \overline{\sigma(\mathcal{H})/\{\lambda\}}$.

If we suppose that $\mathcal{H}$ is bounded then, we define the Kuratowski measure of noncompactness of $\mathcal{H}$ by :

$$\alpha(\mathcal{H}) = \inf\left\{k \in I\!\!R^+ : \ \alpha(\mathcal{H}(B)) \leq k\alpha(B) \text{ for every bounded subset } B \text{ of } Z\right\},$$

where the Kuratowski's measure of noncompactness of the bounded subset B of the Banach space Z is defined by:

$$\alpha(B) = \inf\{d > 0 : B \text{ has a finite cover of diameter } < d\}.$$

The radius of $\sigma_e(\mathcal{H})$ is given by

$$r_e(\mathcal{H}) = \sup\{|\lambda| : \lambda \in \sigma_e(\mathcal{H})\}.$$

Nussbaum has proved in [94] that

$$r_e(\mathcal{H}) = \lim_{n\to+\infty} [\alpha(\mathcal{H}^n)]^{1/n}. \tag{3.7}$$

From the above formula, we deduce that $r_e(\mathcal{H}+\mathcal{K}) = r_e(\mathcal{H})$ for any compact operator $\mathcal{K}: Z \to Z$, that is, r_e is invariant under compact perturbations.

Since $\alpha(\mathcal{H}) \leq \|\mathcal{H}\|$, by the formula (3.7) one has

$$r_e(\mathcal{H}) \leq \|\mathcal{H}\|. \tag{3.8}$$

Let $\lambda \in \sigma(\mathcal{H}) - \sigma_e(\mathcal{H})$, it follows from [122] that λ is an isolate point, $\dim M_\lambda(\mathcal{H}) < \infty$ and $\lambda \in \sigma_p(\mathcal{H})$.

Define the growth bound of the semigroup $(\mathcal{T}(t))_{t\geq 0}$ by

$$\omega((\mathcal{T}(t))_{t\geq 0}) := \inf\left\{\omega \in I\!\!R : \sup_{t\geq 0} e^{-\omega t}\|\mathcal{T}(t)\| < \infty\right\}.$$

Set

$$s_1(A_{\mathcal{T}}) = \sup\left\{\operatorname{Re}\lambda : \lambda \in \sigma(A_{\mathcal{T}}) - \sigma_e(A_{\mathcal{T}})\right\}.$$

It is well-known that in general

$$-\infty \le s_1(A_{\mathcal{T}}) \le \omega((\mathcal{T}(t))_{t\ge 0}) < \infty.$$

Define the essential growth bound of $(\mathcal{T}(t))_{t\ge0}$ by

$$\omega_e((\mathcal{T}(t))_{t\ge 0}) := \lim_{t\to\infty} \frac{1}{t}\log\left(\alpha(\mathcal{T}(t))\right) = \inf\left\{\frac{1}{t}\log\left(\alpha(\mathcal{T}(t))\right) : t > 0\right\}.$$

According to [122], we have

$$\begin{cases} \omega((\mathcal{T}(t))_{t\ge 0}) = \max\left\{\omega_e((\mathcal{T}(t))_{t\ge 0}),\ s_1(A_{\mathcal{T}})\right\}, \\ r_e(\mathcal{T}(t)) = \exp(t\omega_e((\mathcal{T}(t))_{t\ge 0})), \quad t > 0. \end{cases} \tag{3.9}$$

Now, let γ be a positive real constant and

$$C_\gamma = \left\{\phi : (-\infty, 0] \to E \text{ continuous such that } \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) \text{ exists in } E\right\}.$$

It is noted in Chapter 1 that C_γ together with the norm $\|\phi\|_\gamma = \sup_{\theta\le 0} e^{\gamma\theta}|\phi(\theta)|,\ \phi \in C_\gamma,$ is a standard example of phase space $\mathcal{B}$ that satisfies axioms (**A**), (**A1**), (**B**) and (**D1**).

Assume the following hypothesis.

(**H3.5**) There exist two positive constants μ and δ such that

$$\|S'(t)\|_{\overline{D(A)}} \le \mu e^{-\delta t}, \qquad t \ge 0.$$

Proposition 3.4.8 *Suppose that* (**H2.1**)*,* (**H2.2**)*,* (**H3.3**)*,* (**H3.4**) *and* (**H3.5**) *hold, and* $\mathcal{B} = C_\gamma$*. If* $s_1(A_{\mathcal{T}}) < 0$*, then zero is an exponentially stable equilibrium of* $(\mathcal{U}(t))_{t\ge0}$.

Proof. From Proposition 3.4.2, the semigroup $(\mathcal{T}(t))_{t\ge0}$ is decomposed as follows

$$\mathcal{T}(t) = \mathcal{T}_1(t) + \mathcal{T}_2(t),\ t \ge 0,$$

where $(\mathcal{T}_2(t))_{t\ge0}$ is the semigroup on $\mathcal{X}$ given, for $\varphi \in \mathcal{X}$, $t \ge 0$ and $\theta \in (-\infty, 0]$, by

$$\left(\mathcal{T}_2(t)\varphi\right)(\theta) = \begin{cases} S'(t+\theta)\varphi(0), & t+\theta \ge 0, \\ \varphi(t+\theta), & t+\theta \le 0, \end{cases}$$

and $\mathcal{T}_1(t) : \mathcal{X} \to \mathcal{X}$ is compact for $t > 0$. It therefore follows that

$$r_e(\mathcal{T}(t)) = r_e(\mathcal{T}_2(t)).$$

For every $\phi \in \mathcal{X}$, we have

$$\begin{aligned}
\|\mathcal{T}_2(t)\phi\|_\gamma &= \sup_{s\le 0} e^{\gamma s}\left|\left(\mathcal{T}_2(t)\phi\right)(s)\right|, \\
&= \max\left\{\sup_{-t\le s\le 0}\left(e^{\gamma s}\left|S'(t+s)\phi(0)\right|\right),\ \sup_{s\le -t} e^{\gamma s}\left|\phi(t+s)\right|\right\}, \\
&\le \max\left\{\sup_{-t\le s\le 0}\left(\mu e^{\gamma s}e^{-\delta(t+s)}\left|\phi(0)\right|\right),\ \sup_{s\le -t} e^{-\gamma t}e^{\gamma(t+s)}\left|\phi(t+s)\right|\right\}, \\
&\le \mu\max\left\{e^{-\delta t}\sup_{-t\le s\le 0}\left(e^{-(\delta-\gamma)s}\right),\ e^{-\gamma t}\right\}\|\phi\|_\gamma,
\end{aligned}$$

which implies that $\|\mathcal{T}_2(t)\phi\|_\gamma \le \mu e^{-\min(\delta,\gamma)t}\|\phi\|_\gamma$. Using (3.8), we deduce that

$$r_e(\mathcal{T}(t)) \le \mu e^{-\min(\delta,\gamma)t}, \quad t>0.$$

Which implies that for t large enough, $r_e(\mathcal{T}(t)) < 1$, and by (3.9) we get $\omega_e((\mathcal{T}(t))_{t\ge 0}) < 0$. So, we conclude that $\omega((\mathcal{T}(t))_{t\ge 0}) < 0$ and the semigroup $(\mathcal{T}(t))_{t\ge 0}$ is exponentially asymptotically stable. Consequently, from Theorem 3.4.5, we deduce that zero is an exponentially stable equilibrium of $(\mathcal{U}(t))_{t\ge 0}$. This completes the proof of the proposition. ■

3.5 Attractiveness of solutions

The present section gives conditions under which one has existence of a global attractor to Equation (3.2).

The results of this section rely upon the main ideas described by J. Hale in [51] and [56] to give conditions ensuring existence of global attractors for dissipative dynamical systems. Namely, as a consequence of the results obtained in Section 3.4, we give sufficient conditions in order to have the asymptotic smoothness of the semiflow generated by Equation (3.2). Which yields conditions of existence of a global attractor.

For convenience of the reader we recall the following definitions about dissipative equations. Let $(Y, \|.\|)$ be a Banach space.

Definition 3.5.1 *[51] Let* $\mathrm{T}= (T(t))_{t\ge 0}$ *be a (nonlinear) strongly continuous semigroup on* Y.
(i) A set $B\subset Y$ *is said to attract a set* $C\subset Y$ *under* T *if* $dist(T(t)C, B)\to 0$ *as* $t\to +\infty$.
(ii) A set $S\subset Y$ *is said to be invariant under* T *if* $T(t)S = S$ *for all* $t\ge 0$.
(iii) T *is asymptotically smooth if, for any nonempty, closed, bounded set* $B\subset Y$ *for which* $\mathrm{T}B\subset B$, *there is a compact set* $J\subset B$ *such that* J *attracts* B.

(iv) A compact invariant set $\mathcal{A}$ is said to be a maximal compact invariant set if every compact invariant set of the semigroup belongs to $\mathcal{A}$.
(v) An invariant set $\mathcal{A}$ is said to be a global attractor if $\mathcal{A}$ is maximal compact invariant set which attracts each bounded set $B \subset Y$.
(vi) The semigroup T is said to be point dissipative (compact dissipative) if there is a bounded set $B \subset Y$ that attracts each point of Y (each compact set of Y) under T.

Hereafter, we suppose hypothesis (**H2.1**) and (**H3.3**). We also suppose that the initial function φ is an element of

$$\mathcal{X} := \left\{ \phi \in \mathcal{B} : \phi(0) \in \overline{D(A)} \right\}.$$

Remark 3.5.1 *It's easy to see that the assertion* (**v**) *in Proposition 3.4.1 implies that for all $t \geq 0$, $\mathcal{U}(t)$ is a bounded operator on $\mathcal{X}$.*

Proposition 3.5.1 *(Section 3.4) Under the conditions* (**H2.1**), (**H2.2**) *and* (**H3.3**), *let $\mathcal{U}(t) = \mathcal{U}_1(t) + \mathcal{U}_2(t)$, $t \geq 0$, where $(\mathcal{U}_2(t))_{t \geq 0}$ is the solution semigroup of the equation*

$$\begin{cases} x(t) = S'(t)\varphi(0), \ t \geq 0 \\ x_0 = \varphi \in \mathcal{X}. \end{cases}$$

Then $\mathcal{U}_1(t)$ is compact for $t > 0$.

To obtain our main results in this section, we suppose that
(**H3.6**) there exist two positive constants $\bar{M}$ and η such that

$$\|\mathcal{U}_2(t)\| \leq \bar{M} e^{-\eta t}, \ t \geq 0.$$

According to Section 3.4, we can show that the abstract assumption (**H3.6**) is satisfied at least in the case where $\mathcal{B} = C_\gamma$ with $\gamma > 0$. In fact, without loss of generality, we can assume that (**H3.5**) is verified, otherwise, we consider $A - \lambda I$ instead of A such that $S'_\lambda(t) := e^{\lambda t} S'(t)$ is exponentially stable for λ properly chosen (refer to Proposition 1.2.5).

As a consequence of Proposition 3.5.1, we obtain the following result.

Proposition 3.5.2 *Assume that* (**H2.1**), (**H2.2**), (**H3.3**) *and* (**H3.6**) *hold. Then the solution semigroup $(\mathcal{U}(t))_{t \geq 0}$ is asymptotically smooth on $\mathcal{X}$.*

The proof is based on the following lemma.

Lemma 3.5.3 *[51] Suppose that* T *is decomposed as* $T(t) = T_1(t) + T_2(t) : Y \to Y$, $t \geq 0$, *such that* $(T_1(t))_{t\geq 0}$ *is compact and there is a continuous function* $u : I\!R^+ \times I\!R^+ \to I\!R^+$ *such that* $u(t,r) \to 0$ *as* $t \to +\infty$ *and* $\|T_2(t)y\| \leq u(t,r)$ *if* $\|y\| \leq r$. *Then, the semigroup* T *is asymptotically smooth on* Y.

It follows from Hale [51] that

Theorem 3.5.4 *If* (**H2.1**), (**H2.2**), (**H3.3**) *and* (**H3.6**) *hold, then each of the following conditions implies the existence of a global attractor* $\mathcal{A}$ *for Equation (3.2).*
(a) *Equation (3.2) is compact dissipative.*
(b) *Equation (3.2) is point dissipative and orbits of bounded sets are bounded.*

3.6 An application to a reaction diffusion equation with infinite delay

In this section, we apply some of our abstract results to the following reaction diffusion equation with infinite delay

$$\begin{cases} \dfrac{\partial}{\partial t}w(t,\xi) = a\dfrac{\partial^2}{\partial \xi^2}w(t,\xi) + bw(t,\xi) + c\displaystyle\int_{-\infty}^{0} G(\theta)w(t+\theta,\xi)d\theta \\ \qquad\qquad +f(w(t-\tau,\xi)), \qquad\qquad t \geq 0,\ 0 \leq \xi \leq \pi, \\ w(t,0) = w(t,\pi) = 0, \quad t \geq 0, \\ w(\theta,\xi) = w_0(\theta,\xi), \qquad -\infty < \theta \leq 0, \quad 0 \leq \xi \leq \pi. \end{cases} \tag{3.10}$$

where a, b, c and τ are positive constants, $f : I\!R \to I\!R$ is a continuous function, G is a positive integrable function on $(-\infty, 0]$ and $w_0 : (-\infty, 0] \times [0,\pi] \to I\!R$ is an appropriate continuous function.

System (3.10) can be written as the abstract equation (3.2). We choose $E = C([0,\pi]; I\!R)$ and we consider the operator $A : D(A) \subseteq E \to E$ defined by

$$\begin{cases} D(A) = \left\{ y \in C^2([0,\pi]; I\!R) : y(0) = y(\pi) = 0 \right\}, \\ Ay = ay''. \end{cases}$$

It is well-known, see [40], that

$$\begin{cases} (0,+\infty) \subset \rho(A), \\ \left\| (\lambda I - A)^{-1} \right\| \leq \dfrac{1}{\lambda}, \quad \text{for } \lambda > 0. \end{cases}$$

This implies that assumption (**H2.1**) is satisfied. On the other hand, we can see that

$$\overline{D(A)} = \{y \in E : y(0) = y(\pi) = 0\} \neq E.$$

Let, for $\gamma > 0$,

$$C_\gamma = \left\{ \varphi \in C((-\infty, 0]\,; E) : \lim_{\theta \to -\infty} e^{\gamma\theta}\varphi(\theta) \text{ exists in } E \right\}$$

with the norm $\|\phi\|_\gamma = \sup_{\theta \le 0} e^{\gamma\theta} |\phi(\theta)|$, $\phi \in C_\gamma$. As noted in Examples 1.1.1 and 1.1.3, this space satisfies Axioms (**A**), (**A1**), (**B**) and (**D1**). Set

$$\begin{cases} x(t)(\xi) = w(t,\xi), \quad t \ge 0,\ \xi \in [0,\pi]\,, \\ \varphi(\theta)(\xi) = w_0(\theta,\xi), \quad \theta \le 0,\ \xi \in [0,\pi]\,, \\ F(\phi)(\xi) = b\phi(0)(\xi) + f(\phi(-\tau)(\xi)) + c\displaystyle\int_{-\infty}^0 G(\theta)\phi(\theta)(\xi)d\theta, \quad \xi \in [0,\pi]\,,\ \phi \in C_\gamma. \end{cases}$$

Then, Equation (3.10) can be transformed as follows

$$\begin{cases} \dfrac{dx}{dt}(t) = Ax(t) + F(x_t), \qquad t \ge 0, \\ x_0 = \varphi \in C_\gamma. \end{cases}$$

We suppose that

(**i**) f is locally Lipschitz continuous,

(**ii**) $G(.)e^{-\gamma .}$ is integrable on $(-\infty, 0]$,

(**iii**) $w_0 \in C((-\infty,0] \times [0,\pi]\,; I\!R)$, with $\lim_{\theta \to -\infty} \left(e^{\gamma\theta} \sup_{0 \le \xi \le \pi} |w_0(\theta,\xi)| \right)$ exists, and $w_0(0,0) = w_0(0,\pi) = 0$.

We have, for every $\phi_1, \phi_2 \in C_\gamma$,

$$\begin{aligned} &\sup_{0\le\xi\le\pi} \int_{-\infty}^0 G(\theta)\,|\phi_1(\theta)(\xi) - \phi_2(\theta)(\xi)|\,d\theta \\ &= \sup_{0\le\xi\le\pi} \int_{-\infty}^0 e^{-\gamma\theta}G(\theta)\left(e^{\gamma\theta}\,|\phi_1(\theta)(\xi) - \phi_2(\theta)(\xi)|\right)d\theta, \\ &\le \left(\int_{-\infty}^0 e^{-\gamma\theta}G(\theta)d\theta\right) \sup_{\substack{-\infty<\theta\le 0 \\ 0\le\xi\le\pi}} e^{\gamma\theta}\,|\phi_1(\theta)(\xi) - \phi_2(\theta)(\xi)|\,. \end{aligned}$$

Let $r > 0$ and ϕ_1, $\phi_2 \in C_\gamma$ such that $\|\phi_1\|_\gamma\,, \|\phi_2\|_\gamma \le r$.

Then, $|\phi_1(-\tau)(\xi)|\,,\ |\phi_2(-\tau)(\xi)| \le re^{\gamma\tau}$, for every $\xi \in [0,\pi]$. Thus, from assumption (**i**), there exists a positive constant $b(r)$ which depends only on r such that

$$\begin{aligned} \sup_{0\le\xi\le\pi} |f(\phi_1(-\tau)(\xi)) - f(\phi_2(-\tau)(\xi))| &\le b(r) \sup_{0\le\xi\le\pi} |\phi_1(-\tau)(\xi) - \phi_2(-\tau)(\xi)|\,, \\ &\le b(r)e^{\gamma\tau}\,\|\phi_1 - \phi_2\|_\gamma\,. \end{aligned}$$

We conclude that the assumptions (**i**), (**ii**) and (**iii**) imply that F is locally Lipschitz continuous and $\varphi \in C_\gamma$ with $\varphi(0) \in \overline{D(A)}$. Consequently, Theorem 3.2.1 ensures the existence of a maximal interval of existence $(-\infty, b_{w_0})$ and a unique integral solution $w(t,\xi)$ on $(-\infty, b_{w_0}) \times [0,\pi]$.

To investigate that the integral solution w of Equation (3.10) is a strict one, we add the following assumptions.

(**iv**) f is continuously differentiable and f' is locally Lipschitz continuous,

(**v**) $w_0 \in C^2((-\infty,0]\times[0,\pi]; E)$, with $\lim_{\theta\to-\infty}\left(e^{\gamma\theta}\sup_{0\le\xi\le\pi}\left(\frac{\partial}{\partial\theta}w_0(\theta,\xi)\right)\right)$ exists, $\frac{\partial}{\partial\theta}w_0(0,0) = \frac{\partial}{\partial\theta}w_0(0,\pi) = 0$ and

$$\frac{\partial}{\partial\theta}w_0(0,\xi) = a\frac{\partial^2}{\partial\xi^2}w_0(0,\xi) + bw_0(0,\xi)) + c\int_{-\infty}^{0} G(\theta)w_0(\theta,\xi))d\theta + f(w_0(-\tau,\xi)), \text{ for } \xi \in [0,\pi].$$

Then, F is continuously differentiable on C_γ and for $\phi,\psi \in C_\gamma$, $\xi \in [0,\pi]$, we have

$$F'(\phi)(\psi)(\xi) = b\psi(0)(\xi) + c\int_{-\infty}^{0} G(\theta)\psi(\theta)(\xi)d\theta + f'(\phi(-\tau)(\xi))\psi(-\tau)(\xi).$$

F' is also locally Lipschitz continuous on C_γ. Consequently, all the conditions in Theorem 3.3.1 are satisfied. Hence, w is a strict solution of Equation (3.10).

On the other hand, if instead of assumption (**iv**), we assume that

(**vi**) f is differentiable at 0, $f(0)=0$, $f'(0)=0$ and f is globally Lipschitzian.

Then, F is differentiable at 0 with $F(0)=0$ and F is globally Lipschitzian on C_γ. We consider the linearized equation of (3.10) corresponding to the derivative $F'(0)$ at 0 :

$$\begin{cases} \frac{\partial}{\partial t}w(t,\xi) = a\frac{\partial^2}{\partial\xi^2}w(t,\xi) + bw(t,\xi) + c\int_{-\infty}^{0} G(\theta)w(t+\theta,\xi)d\theta, \\ \qquad\qquad t\ge 0,\ 0\le\xi\le\pi, \\ w(t,0) = w(t,\pi) = 0, \quad t\ge 0, \\ w(\theta,\xi) = w_0(\theta,\xi), \quad -\infty<\theta\le 0,\ 0\le\xi\le\pi. \end{cases} \tag{3.11}$$

Let A_0 be the part of the operator A in $\overline{D(A)}$ given by

$$\begin{cases} D(A_0) = \left\{y \in C^2([0,\pi]; I\!R) : y(0) = y''(0) = y(\pi) = y''(\pi) = 0\right\}, \\ A_0 y = ay''. \end{cases}$$

It is known from [43] that A_0 generates a strongly continuous compact and exponentially stable semigroup $(T_0(t))_{t\geq 0}$ in $\overline{D(A)}$.

We deduce from Proposition 3.4.8 that the asymptotic behavior of the linearized semigroup $(\mathcal{T}(t))_{t\geq 0}$ is completely determined by the point spectrum of its generator $A_{\mathcal{T}}$. In order to characterize the eigenvalues of $A_{\mathcal{T}}$, let $\sigma_p(A_{\mathcal{T}})$ denote the point spectrum of $A_{\mathcal{T}}$. Then, $\lambda \in \sigma_p(A_{\mathcal{T}})$ if and only if there exists $\phi \in D(A_{\mathcal{T}})$, $\phi \neq 0$ such that $A_{\mathcal{T}}\phi = \lambda\phi$. It follows that $\phi(\theta) = e^{\lambda\theta}u$ with $u \neq 0$, $u \in D(A)$ and $\lambda u = Au + F'(0)(e^{\lambda\cdot}u)$. On the other hand, $\phi \in C_\gamma$. This is true only if $\operatorname{Re}\lambda \geq -\gamma$. Then, $\lambda \in \sigma_p(A_{\mathcal{T}})$ if and only if there exists $u \in D(A)$ and $u \neq 0$ such that

$$\operatorname{Re}\lambda \geq -\gamma \text{ and } \lambda u = Au + bu + c\int_{-\infty}^{0} G(\theta)e^{\lambda\theta}u d\theta.$$

This means that

$$\operatorname{Re}\lambda \geq -\gamma \text{ and } \lambda - b - c\int_{-\infty}^{0} G(\theta)e^{\lambda\theta}d\theta \in \sigma_p(A) = \sigma_p(A_0).$$

We know that the point spectrum $\sigma_p(A_0)$ of A_0 is given by

$$\sigma_p(A_0) = \left\{-an^2 : n \in I\!N^*\right\}.$$

It follows that the exponential stability of solutions of Equation (3.11) is determined by the following characteristic equation

$$\begin{cases} \operatorname{Re}\lambda \geq -\gamma, \\ \lambda - b - c\displaystyle\int_{-\infty}^{0} G(\theta)e^{\lambda\theta}d\theta = -an^2, \quad n \geq 1. \end{cases} \tag{3.12}$$

In [78], Lenhart and Travis established the following result.

Lemma 3.6.1 *[78] All roots of System (3.12) have negative real part if*

$$\int_{-\infty}^{0} G(\theta)d\theta = 1 \text{ and } c < a - b. \tag{3.13}$$

From this lemma, we conclude that if condition (3.13) is satisfied, then the solution semigroup associated to Equation (3.11) is exponentially stable. In that case, by Proposition 3.4.8 we conclude that zero is a locally exponentially stable equilibrium of the solution semigroup associated to Equation (3.10).

Chapter 4

Existence and Stability for Some Partial Neutral Functional Differential Equations with Infinite Delay[1]

This chapter establishes some extensions to the theory of existence and linearized stability obtained in Chapters 2 and 3.

We start with a brief description of the equation under investigation. Then, in Section 4.2, we prove global existence and uniqueness of integral solutions by use of a fixed point argument. We also prove that the integral solutions are strict solutions under more restrictive assumptions. Next, in Section 4.3, we state some properties of the solution operator associated to the autonomous case. Finally, in Section 4.4, we investigate the stability near an equilibrium. Mainly, we prove that the equilibrium of the solution semigroup associated to the semilinear autonomous case is locally exponentially stable when its linearized solution semigroup around an equilibrium is exponentially stable.

[1] This chapter is based on an ongoing work in collaboration with M. Adimy and K. Ezzinbi.

4.1 Introduction

In [10]-[12], Adimy and Ezzinbi dealed with some general equations of the type (0.14) or (0.15) with finite delay. The authors studied equations of the form (0.15) or :

$$\begin{cases} \dfrac{\partial}{\partial t}\mathcal{D}v_t = A\mathcal{D}v_t + F(v_t), & t \geq 0, \\ v_0 = \varphi \in \mathcal{C}, \end{cases} \tag{4.1}$$

where A is a nondensely defined linear operator that satisfies the Hille-Yosida condition on a Banach space $(E, |.|)$, $\mathcal{C}$ is the space of all continuous functions on $[-r, 0]$ with values in E, provided with the uniform norm topology. $\mathcal{D} : \mathcal{C} \to E$ is a bounded linear operator given by

$$\mathcal{D}\phi := \phi(0) - \int_{-r}^{0} [d\eta(\theta)]\, \phi(\theta),$$

where η is of bounded variation on $[-r, 0]$ and nonatomic at 0. F is a Lipschitz continuous function from $\mathcal{C}$ to E. This last condition implies the well-posedness of Equation (4.1). The authors have established several results concerning the existence and regularity of solutions. They have also obtained several results concerning the stability and the asymptotic behavior of the solution semigroup.

Recently, in [13], the authors have established some results about the local existence and global continuation to Equation (4.1), when F is just completely continuous and the semigroup generated by the part of A in $\overline{D(A)}$ is compact whenever $t > 0$.

However, Hernandez and Henriquez [61] and [62] established some results concerning existence and qualitative properties of solutions to the following PNFDE with infinite delay

$$\begin{cases} \dfrac{d}{dt}[x(t) - G(t, x_t)] = Ax(t) + F(t, x_t), & t \geq 0, \\ x_0 = \varphi \in \mathcal{B}, \end{cases} \tag{4.2}$$

where A generates a strongly continuous semigroup on a Banach space $(E, |.|)$ and $\mathcal{B}$ is the space of functions mapping $(-\infty, 0]$ into E, which satisfies axioms (**A**), (**A1**) and (**B**). G and F are continuous functions from $[0, +\infty) \times \mathcal{B}$ into E and for each $x : (-\infty, b] \to E$, $b > 0$, and $t \in [0, b]$, x_t represents, as usual, the mapping defined from $(-\infty, 0]$ into E by

$$x_t(\theta) = x(t + \theta), \quad \text{for } \theta \in (-\infty, 0].$$

The authors have obtained some results about the existence and regularity by use of the analytic semigroups theory.

In the previous chapters, we had developed many similar results on PFDEs with infinite delay under the condition that A only satisfies the usual Hille-Yosida conditions expect the density of $D(A)$ in E. In this chapter, we prove that the same results can be reproduced for PNFDEs. We consider the case of infinite delay for the equation considered by Adimy and Ezzinbi in [12] :

$$\begin{cases} \dfrac{d}{dt}\left[x(t) - G(t, x_t)\right] = A\left[x(t) - G(t, x_t)\right] + F(t, x_t), \quad t \geq 0, \\ x_0 = \varphi \in \mathcal{B}. \end{cases} \tag{4.3}$$

Equation (4.3) is the true generalization to the infinite delay case of Equation (**??**). However, according to [10], Equation (4.2) can be written as Equation (4.3), if we suppose that $Range(G) \subseteq D(A)$.

In this chapter, we suppose that A is nondensely defined and satisfies the Hille-Yosida condition. As in [10]-[12], we will establish some results about existence of integral and strict solutions to Equation (4.3). In autonomous case, we prove that the solutions define a nonlinear strongly continuous semigroup on $\mathcal{B}$. As a consequence, we give a linearization principle around an equilibrium.

4.2 Existence and regularity of solutions

We first study the existence and uniqueness of solutions to Equation (4.3). We suppose that $(\mathcal{B}, \|.\|_{\mathcal{B}})$ is a seminormed abstract linear space of functions mapping $(-\infty, 0]$ into E, which satisfies the fundamental axioms (**A**), (**A1**) and (**B**). We also assume that A satisfies the condition (**H2.1**). Recall that under this condition**,** A is the generator of a locally Lipschitz continuous integrated semigroup $(S(t))_{t\geq 0}$ on E. It's well known that the derivative $(S'(t))_{t\geq 0}$ of $(S(t))_{t\geq 0}$ generates a C_0-semigroup on $\overline{D(A)}$ such that

$$\left|S'(t)x\right| \leq \bar{M}e^{\omega t}\left|x\right|, \text{ for all } t \geq 0 \text{ and } x \in \overline{D(A)}.$$

We start by introducing the following definitions.

Definition 4.2.1 *Let* $\varphi \in \mathcal{B}$. *We say that a function* $x : (-\infty, a] \to E$, $a > 0$, *is an integral solution of Equation (4.3) in* $(-\infty, a]$ *if the following conditions hold :*
(i) x *is continuous on* $[0, a]$;
(ii) $x(t) = \varphi(t), \quad -\infty < t \leq 0;$
(iii) $\displaystyle\int_0^t (x(s) - G(s, x_s))ds \in D(A)$, *for* $t \in [0, a]$;

(**iv**) *for* $0 \leq t \leq a$,

$$x(t) = G(t, x_t) + \varphi(0) - G(0, \varphi) + A \int_0^t (x(s) - G(s, x_s))ds + \int_0^t F(s, x_s)ds.$$

Definition 4.2.2 *Let $\varphi \in \mathcal{B}$. We say that a function $x : (-\infty, a] \to E$ is a strict solution of Equation (4.3) in $(-\infty, a]$ if the following conditions hold :*
(**i**) $x(t) - G(t, x_t) \in C^1([0, a]\,; E) \cap C([0, a]\,; D(A))$;
(**ii**) *x satisfies Equation (4.3) on $(-\infty, a]$.*

Lemma 4.2.1 *From the closedness property of the operator A, we have the following statements :*
(**i**) *If x is an integral solution of Equation (4.3) in $(-\infty, a]$, then for all $t \in [0, a]$, $x(t) - G(t, x_t) \in \overline{D(A)}$. In particular $\varphi(0) - G(0, \varphi) \in \overline{D(A)}$.*
(**ii**) *If x is an integral solution of Equation (4.3) in $(-\infty, a]$, such that $x(t) - G(t, x_t)$ belongs to $C^1([0, a]\,; E)$ or $C([0, a]\,; D(A))$, then x is also a strict solution of Equation (4.3) in $(-\infty, a]$.*

Proof. (**i**) it suffices to remark that for all $t \in [0, a]$, $h > 0$,

$$x(t) - G(t, x_t) = \lim_{h \to 0} \frac{1}{h} \int_t^{t+h} (x(s) - G(s, x_s))ds \text{ and } \int_t^{t+h} (x(s) - G(s, x_s))ds \in D(A).$$

(**ii**) From Definition 4.2.1, for all $t \in [0, a]$ and $h > 0$,

$$A\frac{1}{h} \int_t^{t+h} (x(s) - G(s, x_s))ds =$$

$$\frac{1}{h} \{x(t + h) - G(t + h, x_{t+h}) - x(t) + G(t, x_t)\} - \frac{1}{h} \int_t^{t+h} F(s, x_s)ds.$$

If $x(s) - G(s, x_s)$ is differentiable, since F is continuous, then the right side of the above equality tends to

$$\frac{d}{dt}(x(t) - G(t, x_t)) - F(t, x_t), \text{ as } h \text{ tends to } 0^+, \text{ and}$$

$$\frac{1}{h} \int_t^{t+h} (x(s) - G(s, x_s))ds \text{ tends to } x(t) - G(t, x_t), \text{ as } h \text{ tends to } 0^+.$$

From the closedness of A, we get $x(t) - G(t,x_t) \in D(A)$ and

$$A(x(t) - G(t,x_t)) = \frac{d}{dt}(x(t) - G(t,x_t)) - F(t,x_t) \quad \text{for } t \in [0,a]\,.$$

From what we deduce that x is a strict solution.
On the other hand, if $x(t)-G(t,x_t)$ belongs to $C([0,a]\,;D(A))$. Again from Definition 4.2.1, for all $t \in [0,a]$ and $h > 0$,
$\frac{1}{h}\left\{x(t+h) - G(t+h, x_{t+h}) - x(t) + G(t,x_t)\right\} =$

$$\frac{1}{h}\int_t^{t+h} A(x(s) - G(s,x_s))ds + \frac{1}{h}\int_t^{t+h} F(s,x_s)ds.$$

Since $A(x(s) - G(s,x_s))$ and F are continuous, the right side of the above equality tends to $A(x(t) - G(t,x_t)) + F(t,x_t)$, as h tends to 0^+. Which implies that $x(t) - G(t,x_t)$ is differentiable at the right in t and

$$\frac{d^+}{dt}(x(t) - G(t,x_t)) = A(x(t) - G(t,x_t)) + F(t,x_t).$$

It's known that when the right derivative is continuous then the C^1 property holds. Then, since $A(x(t) - G(t,x_t)) + F(t,x_t)$ is continuous on $[0,a]$, we conclude that $x(t) - G(t,x_t)$ is differentiable on $[0,a]$ and $\frac{d}{dt}(x(t) - G(t,x_t)) = A(x(t) - G(t,x_t)) + F(t,x_t)$. This finishes the proof of the lemma. ■

To obtain global existence and uniqueness of the integral solution, additionally to (**H2.1**), we make the following hypothesis:
(**H4.1**) $G : [0,+\infty) \times \mathcal{B} \to E$ is continuous and there exists $\alpha_0 > 0$ such that $\alpha_0 K(0) < 1$ and

$$|G(t,\varphi_1) - G(t,\varphi_2)| \le \alpha_0 \left\|\varphi_1 - \varphi_2\right\|_{\mathcal{B}} \quad \text{for } \varphi_1, \varphi_2 \in \mathcal{B} \text{ and } t \ge 0.$$

(**H4.2**) $F : [0,+\infty) \times \mathcal{B} \to E$ is continuous and there exists $\beta_0 > 0$ such that

$$|F(t,\varphi_1) - F(t,\varphi_2)| \le \beta_0 \left\|\varphi_1 - \varphi_2\right\|_{\mathcal{B}} \quad \text{for } \varphi_1, \varphi_2 \in \mathcal{B} \text{ and } t \ge 0.$$

Consider the mapping $\mathcal{G} : [0,+\infty) \times \mathcal{B} \to E$ defined by

$$\mathcal{G}(t,\varphi) \;= \varphi(0) - G(t,\varphi), \quad (t,\varphi) \in [0,+\infty) \times \mathcal{B}.$$

Before stating the results, we first rewrite Equation (4.3) in an integrated form, as follows. For an initial value $\varphi \in \mathcal{B}$ such that $\mathcal{G}(0,\varphi) \in \overline{D(A)}$. From the integrated semigroup theory, we know that $x : (-\infty, +\infty) \to E$ is an integral solution of Equation (4.3) if and only if x solves the following system

$$\begin{cases} \mathcal{G}(t, x_t) = S'(t)\mathcal{G}(0, \varphi) + \dfrac{d}{dt}\displaystyle\int_0^t S(t-s)F(s, x_s)ds, & t \geq 0, \\ x_0 = \varphi. \end{cases} \tag{4.4}$$

Theorem 4.2.2 *Assume that the conditions* (**H2.1**), (**H4.1**) *and* (**H4.2**) *are satisfied. Then, for given* $\varphi \in \mathcal{B}$ *such that* $\mathcal{G}(0, \varphi) \in \overline{D(A)}$, *Equation (4.3) has a unique integral solution* $x(., \varphi)$ *which is the unique solution of Equation (4.4) on* $(-\infty, +\infty)$.

Proof. Let $a > 0$ and $C([0, a]; E)$ be the space of continuous functions from $[0, a]$ to E, provided with the uniform norm topology. Let $\varphi \in \mathcal{B}$ such that $\mathcal{G}(0, \varphi) \in \overline{D(A)}$. Consider the nonempty closed subset defined by

$$Z_a(\varphi) := \{z \in C([0, a]; E) : \ z(0) = \varphi(0)\}.$$

For $z \in Z_a(\varphi)$, we define $\tilde{z} : (-\infty, a] \to E$ by

$$\tilde{z}(t) = \begin{cases} z(t), & t \in [0, a], \\ \varphi(t), & -\infty < t \leq 0. \end{cases}$$

Consider the operator $J : Z_a(\varphi) \longrightarrow Z_a(\varphi)$ defined by

$$(Jz)(t) := G(t, \tilde{z}_t) + S'(t)\mathcal{G}(0, \varphi) + \frac{d}{dt}\int_0^t S(t-s)F(s, \tilde{z}_s)ds.$$

Using the hypothesis and Axiom $(\mathbf{A-iii})$, we can see that for every $z_1, z_2 \in Z_a(\varphi)$ and $t \in [0, a]$,

$$|(Jz_1)(t) - (Jz_2)(t)| \leq \left(\alpha_0 + \beta_0 \bar{M} e^{\omega a} a\right) K_a \left\|z_1 - z_2\right\|_{Z_a(\varphi)}.$$

for suitable $\bar{M} \geq 1$ and $\omega \in I\!R$. Since K is continuous and $\alpha_0 K(0) < 1$, we can choose $a > 0$ small enough such that $\left(\alpha_0 + \beta_0 \bar{M} e^{\omega a} a\right) K_a < 1$.
Then J is a strict contraction in $Z_a(\varphi)$, and the fixed point of J gives a unique integral solution $x(., \varphi)$ on $(-\infty, a]$.
Repeating similar arguments respectively in $[a, 2a]$, ...,$[na, (n+1)a]$, $n \geq 2$, we deduce that the integral solution exists uniquely in $(-\infty, +\infty)$. This ends the proof. ■

The following theorem asserts the strictness of the integral solution under more restrictive conditions. To prove it, we suppose, as in Chapters 2 and 3, that the phase space $\mathcal{B}$ is normed and satisfies Axiom (**C1**) or Axiom (**D1**) and

(**H4.3**) G and F are continuously differentiable and their partial derivatives are locally Lipschitzian with respect to the second argument in the sense that for any compact set $Q \subset [0,\infty) \times \mathcal{B}$, there exists a constant $\beta_1 > 0$ such that

$$\begin{cases} \|D_\varphi F(t,\varphi) - D_\varphi F(t,\psi)\| \leq \beta_1 \|\varphi - \psi\|_{\mathcal{B}}\,, \\ \|D_t F(t,\varphi) - D_t F(t,\psi)\| \leq \beta_1 \|\varphi - \psi\|_{\mathcal{B}}\,, \\ \|D_\varphi G(t,\varphi) - D_\varphi G(t,\psi)\| \leq \beta_1 \|\varphi - \psi\|_{\mathcal{B}}\,, \\ \|D_t G(t,\varphi) - D_t G(t,\psi)\| \leq \beta_1 \|\varphi - \psi\|_{\mathcal{B}} \end{cases} \tag{4.5}$$

for all $(t,\varphi), (t,\psi) \in Q$ and $t \geq 0$, where D_tF, $D_\varphi F$, D_tG and $D_\varphi G$ denote the derivatives with respect to t and φ.

Theorem 4.2.3 *Assume that $\mathcal{B}$ is normed and satifies Axiom* (**C1**) *or Axiom* (**D1**) *and the conditions* (**H2.1**), (**H4.1**), (**H4.2**) *and* (**H4.3**) *hold. Then, for each continuously differentiable function $\varphi \in \mathcal{B}$ such that*

$$\varphi' \in \mathcal{B},\ \mathcal{G}(0,\varphi) \in D(A),\ D_\varphi \mathcal{G}(0,\varphi)\varphi' + D_t\mathcal{G}(0,\varphi) \in \overline{D(A)}$$

$$\textit{and } D_\varphi \mathcal{G}(0,\varphi)\varphi' + D_t\mathcal{G}(0,\varphi) = A\mathcal{G}(0,\varphi) + F(0,\varphi),$$

the integral solution asserted by Theorem 4.2.2, is a strict solution of Equation (4.3).

Proof. Let $a > 0$. Then, by Theorem 4.2.2, we know that Equation (4.3) has a unique integral solution $x := x(.,\varphi)$ which is given by

$$\mathcal{G}(t,x_t) = S'(t)\mathcal{G}(0,\varphi) + \frac{d}{dt}\int_0^t S(t-s)F(s,x_s)ds, \ \text{ for } t \in [0,a]\,. \tag{4.6}$$

By Lemma 4.2.1, it's enough to show that x is continuously differentiable on $[0,a]$.
From Corollary 1.2.2, the assumption $\mathcal{G}(0,\varphi) \in D(A)$ implies that
$S'(t)\mathcal{G}(0,\varphi) = S(t)A\mathcal{G}(0,\varphi) + \mathcal{G}(0,\varphi)$, and Equation (4.6) can be written as

$$\mathcal{G}(t,x_t) = \mathcal{G}(0,\varphi) + S(t)A\mathcal{G}(0,\varphi) + \frac{d}{dt}\int_0^t S(t-s)F(s,x_s)ds. \tag{4.7}$$

Consider the following equation

$$\begin{cases} \dfrac{\partial}{\partial t}\left[D_\varphi \mathcal{G}(t,x_t)y_t + D_t\mathcal{G}(t,x_t)\right] = A\left[D_\varphi \mathcal{G}(t,x_t)y_t + D_t\mathcal{G}(t,x_t)\right] \\ \qquad\qquad\qquad\qquad +D_tF(t,x_t) + D_\varphi F(t,x_t)\,y_t, \qquad t \in [0,a]\,, \\ \\ y_0 = \varphi'. \end{cases} \tag{4.8}$$

Assumptions (**H4.1**) and (**H4.2**) imply respectively that,

$$\|D_\varphi G(t,\psi)\| \le \alpha_0 \quad \text{and} \quad \|D_\varphi F(t,\psi)\| \le \beta_0 \quad \text{for all } \psi\in\mathcal{B} \text{ and } t\ge 0.$$

Then, using the same reasoning as in the proof of Theorem 4.2.2, one can show that Equation (4.8) has a unique integral solution y on $(-\infty,a]$ given by

$$\begin{cases} D_\varphi\mathcal{G}(t,x_t)y_t = -D_t\mathcal{G}(t,x_t) + S'(t)(D_\varphi\mathcal{G}(0,\varphi)\varphi' + D_t\mathcal{G}(0,\varphi)) \\ \qquad\qquad + \dfrac{d}{dt}\displaystyle\int_0^t S(t-s)\left(D_tF(s,x_s) + D_\varphi F(s,x_s)\,y_s\right)ds, \qquad t\in[0,a], \\ y(t) = \varphi'(t), \qquad -\infty < t \le 0. \end{cases} \tag{4.9}$$

Let $w:(-\infty,a]\to E$ be the function defined by

$$w(t) = \begin{cases} \varphi(t) & \text{if } t\in(-\infty,0], \\ \varphi(0) + \displaystyle\int_0^t y(s)\,ds & \text{if } t\in[0,a], \end{cases}$$

then, we can see by use of Lemma 2.2.6 or Lemma 2.2.7 that

$$w_t = \varphi + \int_0^t y_s\,ds, \qquad \text{for } t\in[0,a]. \tag{4.10}$$

Next, we show that $x = w$ on $(-\infty,a]$. As in the proof of the preceding theorem, we proceed by steps, that is, we first take a small enough such that $\alpha_0 K_a < 1$, then we use the same arguments to see the similar result on $[a,2a]$, ..., $[na,(n+1)a]$ for any $n\ge 2$.
Using (4.9) and the expressions satisfied by φ, we obtain for $0\le t\le a$

$$\begin{aligned} \int_0^t D_\varphi\mathcal{G}(s,x_s)y_s\,ds = & -\int_0^t D_t\mathcal{G}(s,x_s)\,ds + S(t)\left(A\mathcal{G}(0,\varphi) + F(0,\varphi)\right) \\ & + \int_0^t S(t-s)\left(D_tF(s,x_s) + D_\varphi F(s,x_s)\,y_s\right)ds. \end{aligned} \tag{4.11}$$

On the other hand, from (4.10), the function $t\mapsto w_t$ is continuously differentiable. It follows that for $t\in[0,a]$, one has

$$\begin{aligned} \frac{d}{dt}\int_0^t S(t-s)F(s,w_s)\,ds = \; & S(t)F(0,\varphi) \\ & + \int_0^t S(t-s)\left(D_tF(s,w_s) + D_\varphi F(s,w_s)\,y_s\right)ds. \end{aligned}$$

Consequently, for $t \in [0,a]$,

$$S(t)\,F(0,\varphi) = \frac{d}{dt}\int_0^t S(t-s)\,F(s,w_s)\,ds - \int_0^t S(t-s)\left(D_tF(s,w_s) + D_\varphi F(s,w_s)\,y_s\right)ds. \tag{4.12}$$

Consider the functions z_1 and z_2 defined on $[0,a]$ by

$$z_1(t) = \mathcal{G}(t,x_t) \quad \text{and} \quad z_2(t) = \mathcal{G}(t,w_t).$$

Using expression (4.7), we get for $t \in [0,a]$,

$$z_1(t) = \mathcal{G}(0,\varphi) + S(t)A\mathcal{G}(0,\varphi) + \frac{d}{dt}\int_0^t S(t-s)\,F(s,x_s)\,ds, \tag{4.13}$$

and

$$\begin{aligned} z_2(t) &= \int_0^t \frac{d}{ds}(\mathcal{G}(s,w_s))ds + \mathcal{G}(0,\varphi), \\ &= \int_0^t \left(D_t\mathcal{G}(s,w_s) + D_\varphi\mathcal{G}(s,w_s)y_s\right)\,ds + \mathcal{G}(0,\varphi). \end{aligned}$$

Then, we infer from expression (4.11)

$$\begin{aligned} z_2(t) = {} & \int_0^t \left(D_t\mathcal{G}(s,w_s) - D_t\mathcal{G}(s,x_s)\right)\,ds \\ & + \int_0^t \left(D_\varphi\mathcal{G}(s,w_s) - D_\varphi\mathcal{G}(s,x_s)\right)y_s\,ds \\ & + \mathcal{G}(0,\varphi) + S(t)\left(A\mathcal{G}(0,\varphi) + F(0,\varphi)\right) \\ & + \int_0^t S(t-s)\left(D_tF(s,x_s) + D_\varphi F(s,x_s)\,y_s\right)\,ds. \end{aligned}$$

Therefore,

$$\begin{aligned} z_1(t) - z_2(t) = {} & \frac{d}{dt}\int_0^t S(t-s)\,F(s,x_s)\,ds - S(t)\,F(0,\varphi) \\ & - \int_0^t \left(D_t\mathcal{G}(s,w_s) - D_t\mathcal{G}(s,x_s)\right)\,ds \\ & - \int_0^t \left(D_\varphi\mathcal{G}(s,w_s) - D_\varphi\mathcal{G}(s,x_s)\right)y_s\,ds \\ & - \int_0^t S(t-s)\left(D_tF(s,x_s) + D_\varphi F(s,x_s)\,y_s\right)\,ds. \end{aligned}$$

Expression (4.12) yields

$$\begin{aligned} z_1(t) - z_2(t) = & \frac{d}{dt}\int_0^t S(t-s)\left(F(s,x_s) - F(s,w_s)\right) ds \\ & -\int_0^t \left(D_t\mathcal{G}(s,w_s) - D_t\mathcal{G}(s,x_s)\right)\ ds \\ & -\int_0^t \left(D_\varphi\mathcal{G}(s,w_s) - D_\varphi\mathcal{G}(s,x_s)\right) y_s\, ds \\ & +\int_0^t S(t-s)\left(D_tF(s,w_s) - D_tF(s,x_s)\right)\ ds \\ & +\int_0^t S(t-s)\left(D_\varphi F(s,w_s) - D_\varphi F(s,x_s)\right) y_s\, ds. \end{aligned}$$

Consequently, we deduce the existence of a positive constant $\sigma(a)$ such that

$$|z_1(t) - z_2(t)| \leq \sigma(a)\int_0^t \|x_s - w_s\|_{\mathcal{B}}\ ds \text{ for } t \in [0,a].$$

Since $x_0 = w_0 = \varphi$, Axiom $(\mathbf{A} - \mathbf{iii})$ implies that

$$\|x_t - w_t\|_{\mathcal{B}} \leq K_a \sup_{0\leq s\leq t} |x(s) - w(s)|,$$

and

$$\begin{aligned} |x(t) - w(t)| & \leq \alpha_0 \|x_t - w_t\|_{\mathcal{B}} + \sigma(a)\int_0^t \|x_s - w_s\|_{\mathcal{B}}\ ds \\ & \leq \alpha_0 K_a \sup_{0\leq s\leq t} |x(s) - w(s)| + \sigma(a)\int_0^t \|x_s - w_s\|_{\mathcal{B}}\ ds. \end{aligned}$$

Therefore, one has

$$\begin{aligned} \|x_t - w_t\|_{\mathcal{B}} & \leq K_a \sup_{0\leq s\leq t} |x(s) - w(s)| \\ & \leq K_a(1-\alpha_0 K_a)^{-1}\sigma(a)\int_0^t \|x_s - w_s\|_{\mathcal{B}}\ ds. \end{aligned}$$

Using the Gronwall lemma, we conclude that

$$\|x_t - w_t\|_{\mathcal{B}} = 0 \quad \text{for any} \quad t \in [0,a].$$

Consequently, $x(t) = w(t)$ for all $t \in (-\infty, a]$, $a = a_0$. Repeating the same procedure in $[a, 2a], ..., [na, (n+1)a]$, $n \geq 2$, we deduce that $x(t) = w(t)$ for all $t \in (-\infty, +\infty)$ and x is continuously differentiable on $(-\infty, +\infty)$. Finally, by Lemma 4.2.1, we get that x is a strict solution. Hence, the proof of Theorem 4.2.3 is complete. ■

4.3 The solution semigroup in autonomous case

In this section, we suppose that F and G are autonomous in t. In that case, Equation (4.3) becomes

$$\begin{cases} \dfrac{d}{dt}\left[x(t) - G(x_t)\right] = A\left[x(t) - G(x_t)\right] + F\left(x_t\right), \quad t \geq 0, \\ x_0 = \varphi \in \mathcal{B}, \end{cases} \tag{4.14}$$

where F and G are Lipschitz continuous on $\mathcal{B}$. We verify that the integral solutions of Equation (4.14) satisfy the properties of a nonlinear strongly continuous semigroup on a subset $\mathcal{Y}$ of $\mathcal{B}$. We also prove that this semigroup satisfies the translation property and a Lipschitz property.

Let $\mathcal{G} : \mathcal{B} \to E$ be the mapping defined by $\mathcal{G}\varphi = \varphi(0) - G\varphi$ and $\mathcal{Y}$ be the set defined by

$$\mathcal{Y} = \left\{\varphi \in \mathcal{B} : \quad \mathcal{G}(\varphi) \in \overline{D(A)}\right\}.$$

For each $t \geq 0$, define the operator $U(t)$ on $\mathcal{Y}$ by

$$U(t)(\varphi) = x_t(., \varphi),$$

where $x(., \varphi)$ is the integral solution of Equation (4.14). From Lemma 4.2.1, it follows that

$$U(t)(\mathcal{Y}) \subseteq \mathcal{Y} \quad \text{for all } t \geq 0.$$

We have the following result:

Proposition 4.3.1 *Assume that the conditions* (**H2.1**), (**H4.1**) *and* (**H4.2**) *hold. Then* $(U(t))_{t\geq 0}$ *is a nonlinear strongly continuous semigroup on* $\mathcal{Y}$*, that is,*
(i) $U(0) = I$*,*
(ii) $U(t+s) = U(t)U(s)$ *for all* $t, s \geq 0$*,*
(iii) for all $\varphi \in \mathcal{Y}$*,* $U(t)(\varphi)$ *is a continuous function of* $t \geq 0$ *with values in* $\mathcal{Y}$*.*
Moreover,
(iv) for all $t \geq 0$*,* $U(t)$ *is continuous from* $\mathcal{Y}$ *into* $\mathcal{Y}$*.*
(v) $(U(t))_{t\geq 0}$ *satisfies for* $t \geq 0$ *and* $\theta \in (-\infty, 0]$ *the translation property*

$$\left(U(t)(\varphi)\right)(\theta) = \begin{cases} \left(U(t+\theta)(\varphi)\right)(0) & \text{if } t+\theta \geq 0, \\ \varphi(t+\theta) & \text{if } t+\theta \leq 0, \end{cases}$$

(vi) for all $T > 0$*, there are two functions* $m, n \in L^\infty([0,T], \mathbb{R}^+)$ *such that for all* $\varphi_1, \varphi_2 \in \mathcal{Y}$*,*

$$\|U(t)\varphi_1 - U(t)\varphi_2\|_{\mathcal{B}} \leq m(t)e^{n(t)} \|\varphi_1 - \varphi_2\|_{\mathcal{B}}, \ t \in [0,T].$$

Proof. The proofs of (i) and (v) are straightforward. (iii) follows from Axiom (**A1**) and the uniqueness of the integral solutions to Equation (4.14). To prove (vi), without loss of generality we suppose that $\omega > 0$. Let $x^1 := x(.,\varphi_1)$ and $x^2 := x(.,\varphi_2)$. Choose $\varepsilon > 0$. For $t \in [0,\varepsilon]$

$$\begin{aligned}
\|U(t)\varphi_1 - U(t)\varphi_2\|_{\mathcal{B}} &= \left\|x_t^1 - x_t^2\right\|_{\mathcal{B}}, \\
&\leq K(t) \sup_{0\leq s\leq t} \left|x^1(s) - x^2(s)\right| + M(t)\left\|\varphi_1 - \varphi_2\right\|_{\mathcal{B}}, \\
&\leq K_\varepsilon \sup_{0\leq s\leq t} \left\{\left|G(x_s^1) - G(x_s^2)\right| + \left|S'(s)(\mathcal{G}(\varphi_1) - \mathcal{G}(\varphi_2))\right|\right\} \\
&\qquad + K_\varepsilon \sup_{0\leq s\leq t} \left|\frac{d}{ds}\int_0^s S(s-\sigma)\left(F(x_\sigma^1) - F(x_\sigma^2)\right) d\sigma\right| \\
&\qquad + M_\varepsilon \left\|\varphi_1 - \varphi_2\right\|_{\mathcal{B}},
\end{aligned}$$

Then

$$\begin{aligned}
\|U(t)\varphi_1 - U(t)\varphi_2\|_{\mathcal{B}} \leq \alpha_0 K_\varepsilon \sup_{0\leq s\leq t} \left\|x_s^1 - x_s^2\right\|_{\mathcal{B}} + \left[K_\varepsilon \bar{M} e^{\omega t}(H + \alpha_0) + M_\varepsilon\right] \left\|\varphi_1 - \varphi_2\right\|_{\mathcal{B}} \\
+ K_\varepsilon \sup_{0\leq s\leq t} \left|\frac{d}{ds}\int_0^s S(s-\sigma)\left(F(x_\sigma^1) - F(x_\sigma^2)\right) d\sigma\right|
\end{aligned}$$

Using Proposition 1.2.9, we have for $0 \leq s \leq t$ and $\lambda > \omega$,

$$\begin{aligned}
&\left|B_\lambda \frac{d}{ds}\int_0^s S(s-\sigma)\left(F(x_\sigma^1) - F(x_\sigma^2)\right) d\sigma\right| \\
&\qquad \leq \left|\int_0^s \|S'(s-\sigma)\| \, \|B_\lambda\| \left|F(x_\sigma^1) - F(x_\sigma^2)\right| d\sigma\right|.
\end{aligned}$$

Since from (**H2.1**),

$$\|B_\lambda\| \leq \frac{\lambda \bar{M}}{\lambda - \omega} \underset{\lambda \to +\infty}{\longrightarrow} \bar{M}, \tag{4.15}$$

by letting λ to $+\infty$, we obtain

$$\begin{aligned}
&\left|\frac{d}{ds}\int_0^s S(s-\sigma)\left(F(x_\sigma^1) - F(x_\sigma^2)\right) d\sigma\right| \\
&\qquad \leq \bar{M}^2 e^{\omega t} \beta_0 \int_0^s \left\|x_\sigma^1 - x_\sigma^2\right\|_{\mathcal{B}} d\sigma.
\end{aligned}$$

Choose $\varepsilon > 0$ such that $1 - K_\varepsilon \alpha_0 > 0$, then for $t \in [0,\varepsilon]$

$$\begin{aligned}
\sup_{0\leq s\leq t} \left\|x_s^1 - x_s^2\right\|_{\mathcal{B}} &\leq (1 - \alpha_0 K_\varepsilon)^{-1} \left\{[K_\varepsilon \bar{M} e^{\omega\varepsilon}(H + \alpha_0)\right. \\
&\quad \left. + M_\varepsilon] \left\|\varphi_1 - \varphi_2\right\|_{\mathcal{B}} + K_\varepsilon \bar{M}^2 e^{\omega\varepsilon} \beta_0 \int_0^t \sup_{0\leq s\leq \sigma} \left\|x_s^1 - x_s^2\right\|_{\mathcal{B}} d\sigma\right\}.
\end{aligned}$$

By use of the Gronwall lemma, we get

$$\sup_{0\leq s\leq t} \left\|x_s^1 - x_s^2\right\|_{\mathcal{B}} \leq v_0(\varepsilon) \left\|\varphi_1 - \varphi_2\right\|_{\mathcal{B}}, \tag{4.16}$$

where

$$\upsilon_0(\varepsilon) = (1-\alpha_0 K_\varepsilon)^{-1}[K_\varepsilon \bar{M} e^{\omega\varepsilon}(H+\alpha_0) + M_\varepsilon]\, exp\left\{(1-\alpha_0 K_\varepsilon)^{-1} K_\varepsilon \bar{M}^2 e^{\omega\varepsilon}\beta_0\varepsilon\right\}.$$

Repeating similar arguments, we obtain similar estimates for $t \in [n\varepsilon, (n+1)\varepsilon]$, $n \geq 2$. Consequently, we get that (vi) is true. Finally, (iv) is an immediate consequence of (vi). This ends the proof. ■

4.4 Linearized stability principle

In this section, we study the stability of an equilibrium of the following autonomous equation

$$\begin{cases} \dfrac{d}{dt}(\mathcal{D}x_t) = A\mathcal{D}x_t + F(x_t), \quad t \geq 0, \\ x_0 = \varphi \in \mathcal{B}, \end{cases} \tag{4.17}$$

where $\mathcal{D}$ satisfies the condition
(**H4.4**) The operator $\mathcal{D} : \mathcal{B} \to E$ is defined by $\mathcal{D}\varphi = \varphi(0) - \mathcal{D}_0\varphi$, for any $\varphi \in \mathcal{B}$, where $\mathcal{D}_0$ is a bounded linear operator from $\mathcal{B}$ into E such that $K(0)\,\|\mathcal{D}_0\| < 1$.

We mean by an equilibrium of Equation (4.17), a $\varphi_0 \in \mathcal{B}$ such that $\varphi_0(\theta) = \varphi(0)$ for all $\theta \leq 0$, $\mathcal{D}\varphi_0 \in D(A)$ and $A\mathcal{D}\varphi_0 + F(\varphi_0) = 0$.
Without loss of generality, we can assume that $\varphi_0 = 0$ and $F(0) = 0$. We suppose that
(**H4.5**) F is Fréchet-differentiable at 0.

Let $L = F'(0)$, then the linearized equation of (4.17) around 0 is of the following form:

$$\begin{cases} \dfrac{d}{dt}(\mathcal{D}x_t) = A\mathcal{D}x_t + Lx_t, \quad t \geq 0, \\ x_0 = \varphi \in \mathcal{B}, \end{cases} \tag{4.18}$$

Let $(U(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ be the semigroups on the space

$$\mathcal{B}_{\mathcal{D}} := \left\{\varphi \in \mathcal{B} : \ \mathcal{D}\varphi \in \overline{D(A)}\right\}.$$

associated, respectively, to the equations (4.17) and (4.18). We have the following result :

Theorem 4.4.1 *Assume that* (**H2.1**), (**H4.2**), (**H4.4**) *and* (**H4.5**) *hold, then for every* $t \geq 0$, *the Fréchet-derivative at zero of* $U(t)$ *is* $T(t)$.

For the proof of this theorem, the following lemma is needed.

Lemma 4.4.2 *Assume that* (**H**4.4) *holds. Let* $\varphi \in \mathcal{B}$ *and* $g : [0, +\infty) \to E$ *be a continuous function. Suppose that there exists a function* $x : (-\infty, +\infty) \to E$ *which is continuous on* $[0, +\infty)$ *and satisfies*

$$\begin{cases} \mathcal{D}x_t = g(t), \quad t \geq 0, \\ x_0 = 0. \end{cases}$$

Then, for each $T > 0$, *there exists a function* $b \in L^\infty([0, T], I\!R^+)$ *such that*

$$\|x_t\|_{\mathcal{B}} \leq b(t) \sup_{0 \leq s \leq t} |g(s)|, \; t \in [0, T]. \tag{4.19}$$

Proof of the lemma. Let $\varepsilon > 0$ small enough. Then, for $t \in [0, \varepsilon]$

$$x(t) = \mathcal{D}_0(x_t) + g(t).$$

By Axiom $(\mathbf{A} - \mathbf{iii})$, for $t \in [0, \varepsilon]$

$$\begin{aligned} \|x_t\|_{\mathcal{B}} &\leq K(t) \sup_{0 \leq s \leq t} |x(s)| \\ &\leq K_\varepsilon \|\mathcal{D}_0\| \sup_{0 \leq s \leq t} \|x_s\|_{\mathcal{B}} + K_\varepsilon \sup_{0 \leq s \leq t} |g(s)|. \end{aligned}$$

We choose ε such that $1 - K_\varepsilon \|\mathcal{D}_0\| > 0$. Then,

$$\|x_t\|_{\mathcal{B}} \leq \sup_{0 \leq s \leq t} \|x_s\|_{\mathcal{B}} \leq b_0(\varepsilon) \sup_{0 \leq s \leq t} |g(s)|,$$

with

$$b_0(\varepsilon) = \frac{K_\varepsilon}{1 - K_\varepsilon \|\mathcal{D}_0\|}.$$

Similarly, by Axiom $(\mathbf{A} - \mathbf{iii})$, for $t \in [\varepsilon, 2\varepsilon]$

$$\begin{aligned} \|x_t\|_{\mathcal{B}} &\leq K(t - \varepsilon) \sup_{\varepsilon \leq s \leq t} |x(s)| + M(t - \varepsilon) \|x_\varepsilon\|_{\mathcal{B}} \\ &\leq K_\varepsilon \|\mathcal{D}_0\| \sup_{0 \leq s \leq t} \|x_s\|_{\mathcal{B}} + K_\varepsilon \sup_{0 \leq s \leq t} |g(s)| \\ &\quad + M(t - \varepsilon) b_0(\varepsilon) \sup_{0 \leq s \leq t} |g(s)| \\ &\leq b_1(\varepsilon) \sup_{0 \leq s \leq t} |g(s)|, \end{aligned}$$

where

$$\begin{aligned} b_1(\varepsilon) &= \frac{K_\varepsilon + M_\varepsilon b_0(\varepsilon)}{1 - K_\varepsilon \|\mathcal{D}_0\|} \\ &= b_0(\varepsilon) + \frac{M_\varepsilon b_0^2(\varepsilon)}{K_\varepsilon}. \end{aligned}$$

Using the same argument, we can see that for $t \in [2\varepsilon, 3\varepsilon]$

$$\begin{aligned} b_2(\varepsilon) &= \frac{K_\varepsilon + M_\varepsilon b_1(\varepsilon)}{1 - K_\varepsilon \|\mathcal{D}_0\|} \\ &= b_0(\varepsilon) + \frac{M_\varepsilon b_0(\varepsilon) b_1(\varepsilon)}{K_\varepsilon} \\ &= b_0(\varepsilon) + \frac{M_\varepsilon b_0^2(\varepsilon)}{K_\varepsilon} + \frac{M_\varepsilon^2 b_0^3(\varepsilon)}{K_\varepsilon^2}, \end{aligned}$$

and inductively, for $t \in [n\varepsilon, (n+1)\varepsilon]$ for any integer n such that $(n+1)\varepsilon \leq T$,

$$\|x_t\|_{\mathcal{B}} \leq b_n(\varepsilon) \sup_{0 \leq s \leq t} |g(s)|, \text{ with}$$

$$b_n(\varepsilon) = b_0(\varepsilon) \sum_{p=0}^{n} \frac{b_0^p(\varepsilon)}{K_\varepsilon^p} M_\varepsilon^p.$$

Then, we obtain the inequality (4.19) for any $T > 0$. This completes the proof of the lemma. ■

Proof of Theorem 4.4.1. It suffices to show that for each $t > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|U(t)(\varphi) - T(t)\varphi\|_{\mathcal{B}} \leq \varepsilon \|\varphi\|_{\mathcal{B}} \text{ for all } \|\varphi\|_{\mathcal{B}} \leq \delta.$$

We have

$$\begin{aligned} \mathcal{D}[U(t)(\varphi) - T(t)\varphi] &= \frac{d}{dt}\int_0^t S(t-s)\left(F(U(s)(\varphi)) - F(T(s)\varphi)\right) ds \\ &+ \frac{d}{dt}\int_0^t S(t-s)\left[F\left(T(s)\varphi\right) - L\left(T(s)\varphi\right)\right] ds. \end{aligned}$$

Let $v : (-\infty, +\infty) \to E$, $w : (-\infty, +\infty) \to E$ and $g : [0, +\infty) \to E$ be defined by

$$v(t) = \begin{cases} \left(U(t)(\varphi)\right)(0), & \text{if } t \geq 0, \\ \varphi(t), & \text{if } t \in (-\infty, 0], \end{cases} \quad w(t) = \begin{cases} \left(T(t)\varphi\right)(0), & \text{if } t \geq 0, \\ \varphi(t), & \text{if } t \in (-\infty, 0], \end{cases}$$

and

$$\begin{aligned} g(t) = &\frac{d}{dt}\int_0^t S(t-s)\left[F\left(T(s)\varphi\right) - L\left(T(s)\varphi\right)\right] ds \\ &+ \frac{d}{dt}\int_0^t S(t-s)\left(F(U(s)(\varphi)) - F(T(s)\varphi)\right) ds. \end{aligned}$$

Then

$$\begin{cases} \mathcal{D}[v_t - w_t] = g(t) & \text{for } t \geq 0, \\ v_0 - w_0 = 0. \end{cases}$$

Using Lemma 4.4.2, we obtain

$$\|v_t - w_t\|_{\mathcal{B}} \leq b(t) \sup_{0\leq s\leq t} |g(s)| .$$

By virtue of the differentiability of F at 0 and (vi) of Proposition 4.3.1, we deduce that for $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\int_0^t e^{-\omega s} |F(T(s)\varphi) - L(T(s)\varphi)\, ds| \leq \varepsilon \|\varphi\|_{\mathcal{B}} , \quad \text{for } \|\varphi\|_{\mathcal{B}} \leq \delta.$$

Consequently,

$$|g(t)| \leq \bar{M} e^{\omega t} \left(\varepsilon \|\varphi\|_{\mathcal{B}} + \beta_0 \int_0^t e^{-\omega s} \|U(s)(\varphi) - T(s)\varphi\|_{\mathcal{B}}\, ds \right)$$

for $\bar{M} \geq 1$ and $\omega \geq 0$ well chosen. It follows that

$$\begin{aligned} &\|U(t)(\varphi) - T(t)\varphi\|_{\mathcal{B}} \leq \\ &\qquad b(t)\bar{M} e^{\omega t} \left(\varepsilon \|\varphi\|_{\mathcal{B}} + \beta_0 \int_0^t e^{-\omega s} \|U(s)(\varphi) - T(s)\varphi\|_{\mathcal{B}}\, ds \right). \end{aligned}$$

By Gronwall's lemma, we obtain

$$\|U(t)(\varphi) - T(t)\varphi\|_{\mathcal{B}} \leq b(t)\bar{M}\varepsilon \|\varphi\|_{\mathcal{B}} \exp\left[\left(b(t)\bar{M}\beta_0 + \omega\right) t\right].$$

We conclude that $U(t)$ is differentiable at 0 and $D_\varphi U(t)(0) = T(t)$ for each $t \geq 0$. ■

Basing on Proposition 4.3.1, Theorem 4.4.1 and Theorem 3.4.4, we obtain a principle of linearized stability to Equation (4.17) as follows.

Theorem 4.4.3 *Under the same assumptions as in Theorem 4.4.1, if the zero equilibrium of $(T(t))_{t\geq 0}$ is exponentially stable, then the zero equilibrium of $(U(t))_{t\geq 0}$ is locally exponentially stable in the sense that there exist $\delta > 0$, $\mu > 0$ and $k \geq 1$ such that*

$$\|U(t)\varphi\|_{\mathcal{B}} \leq k e^{-\mu t} \|\varphi\|_{\mathcal{B}} , \text{ for } t \geq 0 \text{ and } \varphi \in \mathcal{B}_{\mathcal{D}} \text{ with } \|\varphi\|_{\mathcal{B}} < \delta.$$

Chapter 5

Boundedness and Periodicity of Solutions for Some Partial Functional Differential Equations with Infinite Delay[1]

5.1 Introduction

The goal of this chapter is to discuss the existence of periodic solutions to the equation with infinite delay considered in Chapters 2 and 3 :

$$\begin{cases} \dfrac{d}{dt}x(t) = Ax(t) + F(t, x_t), \ t \geq 0, \\ x_0 = \phi \in \mathcal{B}, \end{cases} \tag{5.1}$$

with the same notations as before. Hereafter, the function $F : [0, +\infty) \times \mathcal{B} \to E$ is continuous and p-periodic in t, for some constant $p > 0$.

To our knowledge, existence of periodic solutions to Equation (5.1) with A being densely defined has been investigated by Henriquez [58]-[59], Hino, Murakami and Yoshizawa [67] and recently, Shin and Naito [114]. The authors discussed the problem of existence of solutions defined in the sense of classical semigroups' theory. In [58], existence of periodic solutions was obtained by application of the Sadovskii fixed point theorem on the well-

[1] This chapter is based on a paper in collaboration with R. Benkhalti and K. Ezzinbi. The paper will appear in Journal of Mathematical Analysis and Applications, (2001).

known Poincaré map. In [59], a method of approximation of Equation (5.1) by a family of similar equations with finite delays was introduced. From which some results on existence of periodic solutions and stability were deduced. In [67], assuming the existence of a bounded solution which is BC-uniformly stable, the existence of an almost periodic solution was established under the condition that A generates a compact C_0-semigroup. In [114], Shin and Naito established the existence of periodic solutions to the nonhomogeneous linear case of Equation (5.1). In some of their results, the linear part was assumed to generate a compact C_0-semigroup. The authors obtained several criteria on the existence of periodic solutions. The followed method is based on the perturbation theory of semi-Fredholm operators and a fixed point theorem for affine maps which has been obtained by Chow and Hale [33].

In Chapters 2 and 3, we have considered Equation (5.1) with A being a not necessarily densely defined operator satisfying the Hille-Yosida condition. Precisely, since $D(A)$ is not densely defined, we have addressed the problem of existence, uniqueness, regularity, existence of a global attractor and local stability... by means of the integrated semigroups theory.

It is our aim in this chapter to investigate the existence of periodic solutions for Equation (5.1) with A satisfies the Hille-Yosida condition without being densely defined. Mainly, we recuperate the announced results in [58]. Then, using a key property of the solution operator, we prove the existence of a periodic solution under slightly different conditions. Finally, in the case where $\mathcal{B}$ is a (uniform) fading memory space, we exhibit a Massera type criterion for the nonhomogeneous linear case. Our main results in this chapter would be considered as some extensions of [33], [58] and [114].

Note that, in the previous chapters, we have been considering a variation of constants formula in the sense of integrated semigroups' theory. Here, we use the formula given by Thieme [117] to see that the two formulas are equivalent.

In Section 5.2, we establish some results on existence of periodic solutions in the nonlinear case. However, in Section 5.3, we deal with the nonhomogeneous linear case, we prove the Massera criterion: the existence of a bounded solution implies the existence of a periodic solution. The remaining section is devoted to an application of our abstract results on a partial functional differential equation with infinite delay.

5.2 Existence of periodic solutions in nonlinear case

Suppose that (**H2.1**) is verified and that $(\mathcal{B}, \|.\|_{\mathcal{B}})$ is a seminormed abstract linear space of functions mapping $(-\infty, 0]$ into E, which satisfies the fundamental axioms (**A**), (**A1**) and (**B**) introduced in Chapter 1 and used in the other previous chapters.

Define the part A_0 of A in $\overline{D(A)}$ by

$$\begin{cases} D(A_0) = \left\{ x \in D(A) : Ax \in \overline{D(A)} \right\}, \\ A_0 x = Ax \quad \text{for } x \in D(A_0) \end{cases}$$

Lemma 5.2.1 *[117] A_0 generates a C_0-semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$.*

It is known from [117] and Chapters 2 and 3 that for $\phi \in \mathcal{B}$ such that $\phi(0) \in \overline{D(A)}$, if the associated integral solution $x(., \phi)$ exists, then it is given by the following formula:

$$x(t, \phi) = \begin{cases} T_0(t)\phi(0) + \lim\limits_{\lambda \to +\infty} \displaystyle\int_0^t T_0(t-s) B_\lambda F(s, x_s(., \phi)) ds, & \text{for } t \geq 0, \\ \phi(t), \ \text{for } t \in (-\infty, 0], \end{cases}$$

where $B_\lambda = \lambda R(\lambda, A)$.

Set

$$\mathcal{X} := \left\{ \phi \in \mathcal{B} : \phi(0) \in \overline{D(A)} \right\},$$

and

$$\mathcal{E} := \{\phi \in \mathcal{X} : \text{ Equation (5.1) has a unique integral solution } x(., \phi) \text{ defined on } \mathbb{R}\}.$$

Proposition 5.2.2 *Assume that* (**H2.1**) *holds. If $\phi \in \mathcal{E}$ such that $x_p(., \phi) = \phi$ then $x(., \phi)$ is a p-periodic solution of Equation (5.1).*

Proof. Let $x(.) := x(., \phi)$ and $y(.) := x(. + p)$. Then for $t \geq 0$

$$y(t) = T_0(t+p)\phi(0) + \lim_{\lambda \to +\infty} \int_0^{t+p} T_0(t+p-s) B_\lambda F(s, x_s) ds,$$

which gives that

$$y(t) = T_0(t)T_0(p)\phi(0) + \lim_{\lambda \to +\infty} \int_0^{p} T_0(t+p-s) B_\lambda F(s, x_s) ds$$

$$
\begin{aligned}
&+ \lim_{\lambda\to+\infty} \int_p^{t+p} T_0(t+p-s)B_\lambda F(s,x_s)ds \\
&= T_0(t)\left[T_0(p)\phi(0) + \lim_{\lambda\to+\infty} \int_0^p T_0(p-s)B_\lambda F(s,x_s)ds\right] \\
&+ \lim_{\lambda\to+\infty} \int_0^t T_0(t-s)B_\lambda F(s+p,x_{s+p})ds \\
&= T_0(t)x(p) + \lim_{\lambda\to+\infty} \int_0^t T_0(t-s)B_\lambda F(s,y_s)ds \\
&= T_0(t)y(0) + \lim_{\lambda\to+\infty} \int_0^t T_0(t-s)B_\lambda F(s,y_s)ds.
\end{aligned}
$$

Since $y_0 = x_p = \phi$, by the uniqueness property of $x(.,\phi)$ we have $y(t) = x(t)$ for all $t \in I\!R^+$. Which proves our claim. ■

Define the Poincaré map $P_p : \mathcal{E} \to \mathcal{X}$ by $P_p(\phi) := x_p(.,\phi)$, where $x(.,\phi)$ is the integral solution of Equation (5.1). By Proposition 5.2.2, it suffices to prove that P_p has a fixed point in order to deduce existence of a p-periodic solution to Equation (5.1). To this end, we study the continuity of P_p.

We make the following hypothesis.

(**H5.1**) The semigroup $(T_0(t))_{t\geq 0}$ is compact on $\overline{D(A)}$, this means, $T_0(t)$ is compact in $\overline{D(A)}$ whenever $t > 0$.

Proposition 5.2.3 *Assume that* (**H2.1**) *holds. Then each of the following conditions implies the continuity of* P_p.

(i) (**H2.1**) *and* (**H2.4**) *are satisfied.*

(ii) (**H2.1**), (**H2.3**) *and* (**H5.1**) *are satisfied and for each* $\psi \in \mathcal{E}$, *there exists* $r_\psi > 0$ *such that the set* $\{x_s(.,\phi) : s \in [0,p],\ \phi \in B(\psi, r_\psi)\}$ *is bounded, where*
$B(\psi, r_\psi) := \{\phi \in \mathcal{E} : \|\phi - \psi\|_{\mathcal{B}} \leq r_\psi\}$.

Proof. *(i)* The continuity of P_p follows from Proposition 3.4.1.

Suppose that condition *(ii)* is verified. Let $\psi \in \mathcal{E}$ and $(\psi^k)_k \subset \mathcal{E}$ be a convergent sequence with $\lim\limits_{k\to+\infty} \psi^k = \psi$. Set $x(.) := x(.,\psi)$ and $x^k(.) := x(.,\psi^k)$. We prove that any subsequence $(\eta^n)_n \subset (\psi^n)_n$ has a subsequence $(\eta^{n_k})_k$ such that $P_p\eta^{n_k} \to P_p\psi$, as $k \to +\infty$. From which we will conclude that $P_p\psi^k \to P_p\psi$. In fact, we first prove that $\{x^k(.) : k \in I\!N\}$ is relatively compact in $C([0,p]\,;E)$. Using Axiom $(\mathbf{A}-(\mathbf{ii})')$, we can see that $\left\{\psi^k(0) : k \in I\!N\right\}$ is relatively compact in E. Suppose that $\left\|\psi^k - \psi\right\|_{\mathcal{B}} \leq r_\psi$. It is clear that $\left\{T_0(t)\psi^k(0) : k \in I\!N\right\}$ is relatively compact in E. Let $0 < t \leq p$ and $0 < \varepsilon < t$. Then

$$\begin{aligned}\lim_{\lambda\to+\infty}\int_0^t T_0(t-s)B_\lambda F(s,x_s^k)ds &= T_0(\varepsilon)\lim_{\lambda\to+\infty}\int_0^{t-\varepsilon} T_0(t-\varepsilon-s)B_\lambda F(s,x_s^k)ds \\ &\quad + \lim_{\lambda\to+\infty}\int_{t-\varepsilon}^t T_0(t-s)B_\lambda F(s,x_s^k)ds.\end{aligned}$$

Since $T_0(\varepsilon)$ is compact, there exists a compact set W_ε such that

$$\left\{T_0(\varepsilon)\left(\lim_{\lambda\to+\infty}\int_0^{t-\varepsilon} T_0(t-\varepsilon-s)B_\lambda F(s,x_s^k)ds\right) : k \in I\!N\right\} \subseteq W_\varepsilon.$$

From the boundedness of F, there exists a positive constant a such that

$$\left|\lim_{\lambda\to+\infty}\int_{t-\varepsilon}^t T_0(t-s)B_\lambda F(s,x_s^k)ds\right| \leq a\varepsilon, \text{ uniformly in } k \in I\!N.$$

We deduce that the set $\left\{x^k(t) : k \in I\!N\right\}$ is totally bounded and therefore is relatively compact in E. In order to prove the equicontinuity, let $0 < t_0 < t \leq p$. Then

$$\begin{aligned}\left|x^k(t) - x^k(t_0)\right| &\leq \left|(T_0(t) - T_0(t_0))\psi^k(0)\right| + \left|\lim_{\lambda\to+\infty}\int_{t_0}^t T_0(t-s)B_\lambda F(s,x_s^k)ds\right| \\ &\quad + \left|(T_0(t-t_0) - I)\lim_{\lambda\to+\infty}\int_0^{t_0} T_0(t_0-s)B_\lambda F(s,x_s^k)ds\right|.\end{aligned}$$

The semigroup $(T_0(t))_{t\geq 0}$ is compact for $t > 0$, it follows from [97] that

$$\lim_{t\to t_0}|T_0(t) - T_0(t_0)| = 0,$$

which implies that

$$\lim_{t\to t_0}\left|(T_0(t) - T_0(t_0))\psi^k(0)\right| = 0, \text{ uniformly in } k \in I\!N.$$

On the other hand there exists a positive constant b such that

$$\left|\lim_{\lambda\to+\infty}\int_{t_0}^{t} T_0(t-s)B_\lambda F(s,x_s^k)ds\right| \leq b(t-t_0), \text{ uniformly in } k \in I\!N.$$

Moreover $W_0 = \left\{\lim_{\lambda\to+\infty}\int_0^{t_0} T_0(t_0-s)B_\lambda F(s,x_s^k)ds : k \in I\!N\right\}$ is relatively compact and it's well known that

$$\lim_{h\to 0}(T_0(h) - I)u = 0, \text{ uniformly in } u \in W_0.$$

Which implies that $\lim\limits_{\substack{t\to t_0\\ t>t_0}} \left|x^k(t) - x^k(t_0)\right| = 0$, uniformly in $k \in I\!N$. Using a similar argument for $0 \leq t < t_0 \leq p$, we prove that $\lim\limits_{\substack{t\to t_0\\ t<t_0}} \left|x^k(t) - x^k(t_0)\right| = 0$, uniformly in $k \in I\!N$. It follows that $\left(x^k(.)\right)_{k\in \mathbf{N}}$ is equicontinuous. By Arzéla-Ascoli's theorem, we deduce that $\left\{x^k(.) : k \in I\!N\right\}$ is relatively compact in $C([0,p]\,;E)$. Therefore, if η^n is a subsequence of ψ^k, there exists a subsequence $x(.,\eta^{n_k})$ of $x(.,\eta^n)$ which converges to a certain function $u(.) \in C([0,p]\,;E)$ with $u(0) = \psi(0)$. Set $u(\theta) = \psi(\theta)$ for $\theta < 0$, using Axiom $(\mathbf{A-iii})$, we infer that for all $s \in [0,p]\,, x_s(.,\eta^{n_k}) \to u_s$ as $k \to \infty$. Since $\{x_s(.,\eta^{n_k}) : 0 \leq s \leq p,\ k \in I\!N\}$ is a bounded set of $\mathcal{B}$ and F takes bounded sets into bounded sets, then we have

$$u(t) = T_0(t)\psi(0) + \lim_{\lambda\to+\infty} \int_0^t T_0(t-s)B_\lambda F(s, u_s(.,\psi))ds,\ t \geq 0.$$

As $(\eta^n)_{n\geq 0}$ is an arbitrary subsequence of $(\psi^n)_{n\geq 0}$, this shows that $x_p(.,\psi^n) \to x_p(.,\psi)$ as $n \to +\infty$, and P_p is continuous on $\mathcal{E}$. ■

To our purpose, we assume that

(**H5.2**) there exists a nonempty closed bounded and convex subset $\mathcal{D}$ in $\mathcal{E}$ such that $P_p(\mathcal{D}) \subseteq \mathcal{D}$ and the set $\{x_s(.,\phi) : s \in [0,p],\ \phi \in \mathcal{D}\}$ is bounded.

Theorem 5.2.4 *Assume that* (**H2.1**), (**H2.3**), (**H5.1**) *and* (**H5.2**) *hold. If*

$$\inf_{0<\sigma<p} M(p-\sigma)[HK(\sigma) \sup_{0\leq t\leq \sigma} \|T_0(t)\| + M(\sigma)] < 1, \tag{5.2}$$

then Equation (5.1) has at least one p-periodic solution.

In order to prove this theorem, we recall the following fixed point theorem.

Theorem 5.2.5 *(Sadovskii's theorem [110]) Let Z be a Banach space, D_0 a convex closed and bounded set of Z and $P : D_0 \to D_0$ an α-condensing map, that is P is continuous and for every bounded set $D \subseteq D_0$ with $\alpha(D) > 0$, where $\alpha(.)$ is the Kuratowski's measure of noncompactness,*

$$\alpha(P(D)) < \alpha(D).$$

Then P has at least one fixed point in D_0.

Proof of Theorem 5.2.4**.** It is clear that the condition (ii) in Proposition 5.2.3 is verified for $\mathcal{E} = \mathcal{D}$. Then the map $P_p(\phi) = x_p(.,\phi)$ is continuous from $\mathcal{D}$ to $\mathcal{D}$. Moreover, there exists an induced application $\hat{P}_p : \widehat{\mathcal{D}} \to \widehat{\mathcal{D}}$ that satisfies the condition $\hat{P}_p(\hat{\varphi}) = \widehat{P_p(\varphi)}$ for all $\hat{\varphi} \in \widehat{\mathcal{D}}$ and every $\varphi \in \hat{\varphi}$. We shall prove that $\hat{P}_p$ is condensing. Since for all $\mathcal{C} \subseteq \widehat{\mathcal{D}}$, there exists $\mathcal{V} \subseteq \mathcal{D}$ such that $\mathcal{C} = \widehat{\mathcal{V}}$ and $\alpha(\mathcal{C}) = \alpha(\mathcal{V})$, it suffices to estimate $\alpha(\hat{P}_p(\widehat{\mathcal{G}}))$ only for each subset $\mathcal{G} \subseteq \mathcal{D}$ with $\alpha(\mathcal{G}) > 0$. If we denote $\mathcal{G}[\sigma, p]$, $0 \le \sigma \le p$, the set defined by

$$\mathcal{G}[\sigma, p] := \left\{ x(.,\varphi)_{/[\sigma,p]} : \varphi \in \mathcal{G} \right\},$$

then, using (**H5.2**) and Axiom ($\mathbf{A} - \mathbf{ii}$), we can see that the conditions of Theorem 2.1 in [112] are verified. Hence

$$\alpha(\hat{P}_p(\widehat{\mathcal{G}})) \le K(p-\sigma)\alpha(\mathcal{G}[\sigma,p]) + M(p-\sigma)\alpha(\widehat{\mathcal{G}}_\sigma), \tag{5.3}$$

where $\mathcal{G}_\sigma = \{x_\sigma(.,\varphi) : \varphi \in \mathcal{G}\}$.
On the other hand, for all $\sigma > 0$, $\mathcal{G}[\sigma, p]$ is relatively compact in $C([\sigma, p]; E)$. To prove this assertion, we use the fact that $T_0(t)$ is compact for any $t > 0$ and we can proceed as in the proof of Proposition 4 in [5]. That is, we prove that the family $\mathcal{G}[\sigma, p]$ is equicontinuous and $\mathcal{G}[\sigma, p](t)$, for each $t \in [0, p]$, is relatively compact in E.
Thus, (5.3) becomes

$$\alpha(\hat{P}_p(\widehat{\mathcal{G}})) \le M(p-\sigma)\alpha(\widehat{\mathcal{G}}_\sigma). \tag{5.4}$$

To estimate $\alpha(\widehat{\mathcal{G}}_\sigma)$, we take $[0, \sigma]$ instead of $[\sigma, p]$ in (5.3), we infer

$$\alpha(\widehat{\mathcal{G}}_\sigma) \le K(\sigma)\alpha(\mathcal{G}[0,\sigma]) + M(\sigma)\alpha(\widehat{\mathcal{G}}). \tag{5.5}$$

Next we prove that

$$\alpha(\mathcal{G}[0,\sigma]) \le H \sup_{0 \le t \le \sigma} \|T_0(t)\| \, \alpha(\mathcal{G}). \tag{5.6}$$

In fact, if $\mathcal{G}(0) = \{\varphi(0) : \varphi \in \mathcal{G}\}$ then Axiom ($\mathbf{A} - \mathbf{ii}$) gives

$$\alpha(\mathcal{G}(0)) \le H\alpha(\mathcal{G}). \tag{5.7}$$

Let $d > 0$ such that $\alpha(\mathcal{G}(0)) < d$. By definition of $\alpha(.)$, $\mathcal{G}(0)$ has a finite cover $(C_i)_{1 \le i \le n} \subseteq E$, such that $diam(C_i) < d$. Henceforth, for a subset C of E, we denote by C^* the set

$$C^* = \left\{ T_0(.)u_{/[0,\sigma]} : u \in C \right\},$$

then it is clear that

$$diam(C_i^*) < \sup_{0 \le t \le \sigma} \|T_0(t)\| \, d$$

and

$$\mathcal{G}(0)^* \subseteq \bigcup_{i=1}^{n} C_i^*.$$

Hence, we deduce that

$$\alpha(\mathcal{G}(0)^*) < \sup_{0\le t\le\sigma} \|T_0(t)\|\, d.$$

From this together with (5.7) and the choice $d := H\alpha(\mathcal{G}) + \varepsilon$, $\varepsilon > 0$, we infer that

$$\alpha(\mathcal{G}(0)^*) \le H \sup_{0\le t\le\sigma} \|T_0(t)\|\, \alpha(\mathcal{G}).$$

Since

$$\mathcal{G}\,[0,\sigma] \subseteq \mathcal{G}(0)^* + \left\{ w(.,\varphi)_{|[0,\sigma]} : \varphi \in \mathcal{G} \right\},$$

where $w(t,\varphi) = \lim_{\lambda\to+\infty} \int_0^t T_0(t-s)B_\lambda F(s, x_s(.,\varphi))ds$, for $t \in [0,\sigma]$. Using similar arguments as in the proof of Proposition 5.2.3, we can see that $\left\{ w(.,\varphi)_{|[0,\sigma]} : \varphi \in \mathcal{G} \right\}$ is relatively compact in $C([0,\sigma]\,; E)$. In fact, let $0 < t \le \sigma$ and $0 < \varepsilon < t$. Then

$$\begin{aligned} &\lim_{\lambda\to+\infty} \int_0^t T_0(t-s)B_\lambda F(s, x_s(.,\varphi))ds \\ &\quad = T_0(\varepsilon) \lim_{\lambda\to+\infty} \int_0^{t-\varepsilon} T_0(t-\varepsilon-s)B_\lambda F(s, x_s(.,\varphi))ds \\ &\qquad + \lim_{\lambda\to+\infty} \int_{t-\varepsilon}^{t} T_0(t-s)B_\lambda F(s, x_s(.,\varphi))ds. \end{aligned}$$

Since $T_0(\varepsilon)$ is compact, there exists a compact set W_ε such that

$$\left\{ T_0(\varepsilon)\left(\lim_{\lambda\to+\infty} \int_0^{t-\varepsilon} T_0(t-\varepsilon-s)B_\lambda F(s, x_s(.,\varphi))ds \right) : \varphi \in \mathcal{G} \right\} \subseteq W_\varepsilon.$$

From the boundedness of F, there exists a positive constant a such that

$$\left| \lim_{\lambda\to+\infty} \int_{t-\varepsilon}^{t} T_0(t-s)B_\lambda F(s, x_s(.,\varphi))ds \right| \le a\varepsilon, \text{ uniformly in } \varphi \in \mathcal{G}.$$

We deduce that the set $\left\{ w(.,\varphi)_{|[0,\sigma]} : \varphi \in \mathcal{G} \right\}$ is totally bounded and therefore is relatively compact in E. To prove the equicontinuity, let $0 < t_0 < t \le \sigma$. Then

$$\begin{aligned} |w(t,\varphi) - w(t_0,\varphi)| \le &\left| \lim_{\lambda\to+\infty} \int_{t_0}^{t} T_0(t-s)B_\lambda F(s, x_s(.,\varphi))ds \right| \\ &+ \left| (T_0(t-t_0) - I) \lim_{\lambda\to+\infty} \int_0^{t_0} T_0(t_0-s)B_\lambda F(s, x_s(.,\varphi))ds \right|. \end{aligned}$$

There exists a positive constant b such that

$$\left| \lim_{\lambda \to +\infty} \int_{t_0}^{t} T_0(t-s) B_\lambda F(s, x_s(., \varphi)) ds \right| \leq b(t-t_0), \text{ uniformly in } \varphi \in \mathcal{G}.$$

Moreover, $W_0 = \left\{ \lim_{\lambda \to +\infty} \int_0^{t_0} T_0(t_0-s) B_\lambda F(s, x_s(., \varphi)) ds : \varphi \in \mathcal{G} \right\}$ is relatively compact and it's well known that

$$\lim_{h \to 0} (T_0(h) - I)u = 0, \text{ uniformly in } u \in W_0.$$

Which implies that $\lim_{\substack{t \to t_0 \\ t > t_0}} |w(t, \varphi) - w(t_0, \varphi)| = 0$, uniformly in $\varphi \in \mathcal{G}$. Using a similar argument for $0 \leq t < t_0 \leq \sigma$, we prove that $\lim_{\substack{t \to t_0 \\ t < t_0}} |w(t, \varphi) - w(t_0, \varphi)| = 0$, uniformly in $\varphi \in \mathcal{G}$. It follows that $\left\{ w(., \varphi)_{|[0,\sigma]} : \varphi \in \mathcal{G} \right\}$ is equicontinuous. By Arzéla-Ascoli's theorem, we deduce that $\left\{ w(., \varphi)_{|[0,\sigma]} : \varphi \in \mathcal{G} \right\}$ is relatively compact in $C([0, \sigma]; E)$. We complete the proof of (5.6) by the properties of $\alpha(.)$.

Thus the inequalities (5.4), (5.5) and (5.6) prove that for all $0 < \sigma \leq p$,

$$\alpha(\hat{P}_p(\widehat{\mathcal{G}})) \leq (M(p-\sigma)[HK(\sigma) \sup_{0 \leq t \leq \sigma} \|T_0(t)\| + M(\sigma)]\alpha(\widehat{\mathcal{G}}).$$

Finally, condition (5.2) implies that $\hat{P}_p$ is α-condensing.
By Theorem 5.2.5, we conclude that there exists $\hat{\varphi} \in \widehat{\mathcal{D}}$ such that $\hat{P}_p \hat{\varphi} = \hat{\varphi}$.
Then $\widehat{P_p^n \varphi} = \hat{P}_p^n(\hat{\varphi}) = \hat{\varphi}$ for all $n \in I\!N$. Since

$$\left\| P_p^m \varphi - P_p^n \varphi \right\|_{\mathcal{B}} = \left\| \widehat{P_p^m \varphi} - \widehat{P_p^n \varphi} \right\|_{\mathcal{B}}, \text{ for } m, n \in I\!N,$$

$(P_p^n \varphi)_n$ is a Cauchy sequence of $\mathcal{B}$. Which implies that there exists $\psi \in \mathcal{D}$ such that $P_p^k \varphi \to \psi$, as $k \to +\infty$. Consequently $P_p \psi = \psi$. This completes the proof by use of Proposition 5.2.2. ■

Remark 5.2.1 *Assumption* (**H5.2**) *seems to be strong, but in nonhomogeneous linear case, we will prove that the existence of a bounded solution implies that* (**H5.2**) *is true.*

In the sequel, we suppose that (**H2.4**) is satisfied.
Define $\mathcal{U}(t)$ on $\mathcal{X}$ for $t \geq 0$ by

$$\mathcal{U}(t)\phi = x_t(., \phi),$$

where $x(.,\phi)$ is the integral solution of Equation (5.1).

The following result on the solution operator $\mathcal{U}(t)$, $t \geq 0$, is a key property to the existence of periodic solutions.

Proposition 5.2.6 *Assume that* $(\mathbf{H2.1})$, $(\mathbf{H2.4})$ *and* $(\mathbf{H5.1})$ *hold. Then* $\mathcal{U}(t)$ *is decomposed as follows*

$$\mathcal{U}(t) = \mathcal{U}_1(t) + \mathcal{U}_2(t), \ t \geq 0,$$

where $(\mathcal{U}_1(t))_{t\geq 0}$ *is the solution semigroup of the following equation*

$$\begin{cases} x(t) = T_0(t)\phi(0), \ t \geq 0 \\ x_0 = \phi \in \mathcal{X}, \end{cases} \tag{5.8}$$

and $\mathcal{U}_2(t)$ *is compact for* $t > 0$.

Proof. We use similar arguments as in Chapter 3. Let $(\psi_n)_{n\geq 0}$ be a bounded sequence in $\mathcal{X}$. Then for $t > 0$ one has

$$(\mathcal{U}_2(t)\psi_n)(\theta) = \begin{cases} (\mathcal{U}_2(t+\theta)\psi_n)(0) \text{ if } \theta \in [-t, 0] \\ 0 \text{ if } \theta \in (-\infty, -t). \end{cases}$$

In order to prove that for all $\theta \in (-\infty, 0]$, $\{(\mathcal{U}_2(t)\psi_n)(\theta) : n \geq 0\}$ is relatively compact in E. Let $0 < t + \theta$ and choose ε such that $\varepsilon < t + \theta$. Then

$$\begin{aligned} (\mathcal{U}_2(t)\psi_n)(\theta) &= \lim_{\lambda\to+\infty} \int_0^{t+\theta} T_0(t+\theta-s)B_\lambda F(s, x_s(.,\psi_n))ds \\ &= T_0(\varepsilon) \lim_{\lambda\to+\infty} \int_0^{t+\theta-\varepsilon} T_0(t+\theta-\varepsilon-s)B_\lambda F(s, x_s(.,\psi_n))ds \\ &\quad + \lim_{\lambda\to+\infty} \int_{t+\theta-\varepsilon}^{t+\theta} T_0(t+\theta-s)B_\lambda F(s, x_s(.,\psi_n))ds. \end{aligned}$$

By Proposition 3.4.1, $\{F(s, x_s(.,\psi_n)) : s \in [0,t], n \geq 0\}$ is bounded. Since $T_0(\varepsilon)$ is compact, there exists a compact set W_ε such that

$$T_0(\varepsilon)\left\{\lim_{\lambda\to+\infty} \int_0^{t+\theta-\varepsilon} T_0(t+\theta-\varepsilon-s)B_\lambda F(s, x_s(.,\psi_n))ds : n \geq 0\right\} \subseteq W_\varepsilon.$$

Furthermore, there exists a positive constant c such that

$$\left|\lim_{\lambda\to+\infty} \int_{t+\theta-\varepsilon}^{t+\theta} T_0(t+\theta-s)B_\lambda F(s, x_s(.,\psi_n))ds\right| \leq c\varepsilon, \text{ uniformly in } n \in \mathbb{N}.$$

Which gives that $\{(\mathcal{U}_2(t)\psi_n)(\theta) : n \geq 0\}$ is totally bounded and therefore relatively compact. To establish the second assertion, it is sufficient to show that $(\mathcal{U}_2(t)\psi_n)_{n\geq 0}$ is equicontinuous in $(-\infty, 0]$. Let $\theta_0 \in (-\infty, 0]$. For $\theta \in (-\infty, 0]$ enough close to θ_0 such that $\theta_0 < \theta$, we see that

$$(\mathcal{U}_2(t)\psi_n)(\theta) - (\mathcal{U}_2(t)\psi_n)(\theta_0) = \begin{cases} (\mathcal{U}_2(t+\theta)\psi_n)(0) - (\mathcal{U}_2(t+\theta_0)\psi_n)(0) \text{ if } \theta_0 > -t \\ (\mathcal{U}_2(t+\theta)\psi_n)(0) \text{ if } \theta_0 = -t \\ 0 \text{ if } \theta_0 < -t. \end{cases}$$

For $-t < \theta_0 < \theta \leq 0$, one has

$$\begin{aligned} &(\mathcal{U}_2(t)\psi_n)(\theta) - (\mathcal{U}_2(t)\psi_n)(\theta_0) \\ &= \lim_{\lambda\to+\infty} \int_0^{t+\theta_0} (T_0(t+\theta-s) - T_0(t+\theta_0-s))\, B_\lambda F(s, x_s(.,\psi_n))ds \\ &\quad + \lim_{\lambda\to+\infty} \int_{t+\theta_0}^{t+\theta} T_0(t+\theta-s) B_\lambda F(s, x_s(.,\psi_n))ds. \end{aligned}$$

By the boundedness of $\{F(s, x_s(.,\psi_n) : n \geq 0, s \in [t+\theta_0, t]\}$, we deduce that there exists a positive constant d such that

$$\begin{aligned} &|(\mathcal{U}_2(t)\psi_n)(\theta) - (\mathcal{U}_2(t)\psi_n)(\theta_0)]| \\ &\leq \left|(T_0(\theta-\theta_0) - I) \lim_{\lambda\to+\infty} \int_0^{t+\theta_0} T_0(t+\theta_0-s) B_\lambda F(s, x_s(.,\psi_n))ds\right| \\ &\quad + d(\theta-\theta_0), \text{ for all } n \geq 0 \end{aligned}$$

On the other hand $W_1 = \left\{ \lim_{\lambda\to+\infty} \int_0^{t+\theta_0} T_0(t+\theta_0-s) B_\lambda F(s, x_s(.,\psi_n))ds : n \geq 0 \right\}$ is relatively compact in E, using the fact that $(T_0(.)x)_{x\in W_1}$ is equicontinuous at the right in 0, we get

$$\lim_{\substack{\theta\to\theta_0 \\ \theta>\theta_0}} |(\mathcal{U}_2(t)\psi_n)(\theta) - (\mathcal{U}_2(t)\psi_n)(\theta_0)| = 0, \text{ uniformly in } n \in I\!N.$$

By a similar argument as above we prove

$$\lim_{\substack{\theta\to\theta_0 \\ \theta<\theta_0}} |(\mathcal{U}_2(t)\psi_n)(\theta) - (\mathcal{U}_2(t)\psi_n)(\theta_0)| = 0, \text{ uniformly in } n \in I\!N.$$

We deduce the claimed equicontinuity. Proceeding by Arzéla-Ascoli's theorem, there is a continuous function $\phi : (-\infty, 0] \to E$ and a subsequence ϕ_n of $(U_2(t)\psi_n)_{n\geq 0}$ which converges compactly to ϕ in $(-\infty, 0]$. In addition ϕ is continuous and $\phi(\theta) = 0$ if $\theta \leq -t$, because $\phi_n(\theta) = 0$ for $\theta \leq -t$ and $n \geq 0$. Hence ϕ belongs to $\mathcal{B}$ by Axiom $(\mathbf{A} - \mathbf{i})$, and moreover we get $\|\phi_n - \phi\|_{\mathcal{B}} \to 0$ as $n \to \infty$ because of $\|\phi_n - \phi\|_{\mathcal{B}} \leq K(t) \sup\limits_{-t \leq \theta \leq 0} |\phi_n(\theta) - \phi(\theta)|$ by Axiom $(\mathbf{A} - \mathbf{iii})$. In definitive, we have proved that the image of any bounded sequence contains a converging subsequence in $\mathcal{B}$ with respect to the seminorm. ■

We need to suppose that

(**H5.3**) There exists a nonempty closed bounded and convex subset $\mathcal{D}$ in $\mathcal{X}$ such that $P_p(\mathcal{D}) \subseteq \mathcal{D}$.

Theorem 5.2.7 *Assume that* (**H2.1**), (**H2.4**), (**H5.1**) *and* (**H5.3**) *hold. If there exists a positive constant* η *such that*

$$\|\mathcal{U}_1(t)\| \leq e^{-\eta t}, \ t \geq 0. \tag{5.9}$$

Then Equation (5.1) has at least one p-periodic solution.

Proof. As in the proof of Theorem 5.2.4, we use Sadovskii's fixed point theorem. Since $\mathcal{U}_2(p)$ is compact, for each subset $\mathcal{G} \subseteq \mathcal{D}$ with $\alpha(\mathcal{G}) > 0$, one has

$$\begin{aligned} \alpha(\widehat{P_p}(\widehat{\mathcal{G}})) = \alpha(\widehat{\mathcal{U}}(p)\widehat{\mathcal{G}}) &= \alpha(\widehat{\mathcal{U}_1}(p)\widehat{\mathcal{G}}) \\ &\leq e^{-\eta p}\alpha(\widehat{\mathcal{G}}) \\ &< \alpha(\widehat{\mathcal{G}}). \end{aligned}$$

From what we deduce that $\widehat{P_p}$ is α-condensing on $\widehat{\mathcal{D}}$, and Equation (5.1) has a p-periodic solution. ■

In relation with the condition (5.9), we recall the following result.

Lemma 5.2.8 *(Chapters 1 and 3) The space* C_γ, $\gamma > 0$ *defined in the previous chapters with the norm* $\|\phi\|_\gamma := \sup\limits_{\theta \leq 0} e^{\gamma\theta} |\phi(\theta)|$, $\phi \in C_\gamma$, *satisfies Axioms* (**A**), (**A1**), *and* (**B**). *Moreover, if* $\|T_0(t)\| \leq e^{-\delta t}$, $t \geq 0$, *where* δ *is a positive constant. Then Condition (5.9) is true.*

In the remaining of this section, we give some sufficient conditions ensuring assumption (**H5.2**) and, in particular, (**H5.3**). Consider the phase space $C_r \times L^1(g)$, $r > 0$, defined in Chapter 1, with the seminorm

$$\|\varphi\|_{\mathcal{B}} := \sup_{-r \leq \theta \leq 0} |\varphi(\theta)| + \int_{-\infty}^{-r} g(\theta) |\varphi(\theta)| \, d\theta.$$

From Chapter 1, if we suppose that g satisfies :
(i) g is integrable on $(-d,-r)$ for any $d \geq r$, and
(ii) there exists a locally bounded function $G : (-\infty, 0] \to [0,+\infty)$ such that

$$g(\xi+\theta) \leq G(\xi)g(\theta), \text{ for all } \xi \leq 0 \text{ and } \theta \in (-\infty,-r) \setminus N_\xi,$$

where $N_\xi \subseteq (-\infty,-r)$ is a set with Lebesgue measure 0,
then $C_r \times L^1(g)$ verifies Axioms (**A**), (**A1**) and (**B**).
Define $\rho(t)$, $t \geq 0$, by

$$\rho(t) := \sup_{\theta \leq -r} \frac{g(\theta - t)}{g(\theta)} + \rho_0(t) + \int_{-t-r}^{-\bar{t}} g(\theta)d\theta$$

where $\bar{t} = \max\{r,t\}$ and $\rho_0 := \mathbf{1}_{(-\infty,r]}$.
In addition, we suppose that $\rho(t) \to 0$, as $t \to +\infty$.

Proposition 5.2.9 *Assuming that $F : [0,+\infty) \times (C_r \times L^1(g)) \to E$ is continuous and locally Lipschitzian with respect to the second variable and there exist $N_1, N_2 > 0$*

$$|F(t,\varphi)| \leq N_1 \|\varphi\|_{\mathcal{B}} + N_2, \ \varphi \in C_r \times L^1(g). \tag{5.10}$$

If there exists $\omega_0 > 0$ and $k_0 > 0$ such that the following conditions hold :
(i) $\|T_0(t)\| \leq \exp(-\omega_0 t)$, $t \geq 0$,
(ii) $\int_0^{t-r} g(s-t)\exp(\omega_0(t-r-s))ds \leq k_0$ for $t \geq r$,
(iii) $N_1 \exp(\omega_0 r)(1+k_0) < \omega_0$.
Then there exists $R > 0$ and $\sigma > 0$ such that $P_\sigma(B(0,R)) \subseteq B(0,R)$ and the set $\{x_t(.,\phi) : \phi \in B(0,R),\ 0 \leq t \leq \sigma\}$ is bounded in $C_r \times L^1(g)$.

The proof of this proposition is omitted here, it can be done similarly as in [58].

5.3 A Massera type criterion in nonhomogeneous case

In this section, we study the nonhomogeneous linear case :

$$\begin{cases} \dfrac{d}{dt}x(t) = Ax(t) + L(t,x_t) + f(t), \ t \geq 0, \\ x_0 = \phi \in \mathcal{B}, \end{cases} \tag{5.11}$$

where L is a continuous function form $\mathbb{R}^+ \times \mathcal{B}$ into E, linear with the second argument and p-periodic in t, f is a continuous p-periodic function.

For $\varphi \in \mathcal{B}$, $t \geq 0$ and $\theta \leq 0$, we define

$$[W(t)\varphi](\theta) = \begin{cases} \varphi(0) \text{ if } t+\theta \geq 0 \\ \varphi(t+\theta) \text{ if } t+\theta < 0. \end{cases}$$

We can see that $(W(t))_{t\geq 0}$ is a C_0-semigroup on $\mathcal{B}$. We set

$$W_0(t) = W(t)_{/\mathcal{B}_0}, \text{ where } \mathcal{B}_0 := \{\varphi \in \mathcal{B} : \varphi(0) = 0\}.$$

Let C_{00} be the set of continuous functions $\varphi : (-\infty, 0] \to E$ with compact support and recall that a sequence of functions $(\varphi^n)_{n\in \boldsymbol{N}} : (-\infty, 0] \to E$, is uniformly bounded if $\sup_{n\in \boldsymbol{N}} \left(\sup_{-\infty<\theta\leq 0} |\varphi^n(\theta)| \right) < +\infty.$

We suppose that $\mathcal{B}$ is a fading memory space in the sense that $\mathcal{B}$ satisfies the additional axioms

(**C**) If a uniformly bounded sequence $(\varphi^n)_n$ in C_{00} converges to a function φ compactly on $(-\infty, 0]$, then φ is in $\mathcal{B}$ and $\|\varphi^n - \varphi\|_{\mathcal{B}} \to 0$ as $n \to +\infty$.

(**D2**) $\|W_0(t)\varphi\|_{\mathcal{B}} \to 0$ as $t \to +\infty$ for all $\varphi \in \mathcal{B}_0$.

It's well known by the Banach Steinhaus theorem that (**D2**) implies $\sup_{t\geq 0} \|W_0(t)\| < \infty$.

Now assume that

(**H5.4**) Equation (5.11) has a bounded solution y on $\mathbb{R}^+$ in the sense that $\sup_{t\in \boldsymbol{R}^+} |y(t)| < +\infty$.

As a consequence of Proposition 1.1.2, if $\mathcal{B}$ is a fading memory space, then the functions $K(.)$ and $M(.)$ can be chosen bounded on $[0, +\infty)$. Hence, using Axiom (**A** $-$ **iii**), we obtain the existence of a positive constant N_1 such that

$$\sup_{t\geq 0} \|y_t\|_{\mathcal{B}} \leq N_1. \tag{5.12}$$

Theorem 5.3.1 *Assume that* (**H2.1**), (**H5.1**), (**H5.4**), (**C**) *and* (**D2**) *are satisfied. If*

$$\inf_{0<\sigma<p} M(p-\sigma)[HK(\sigma) \sup_{0\leq t\leq \sigma} \|T_0(t)\| + M(\sigma)] < 1, \tag{5.13}$$

then Equation (5.11) has a p-periodic solution.

Proof. Set $\mathcal{D} := \overline{co}\{y_{np} : n \in \mathbb{N}\}$ where $\overline{co}$ denotes the closure of the convex hull. Then $\mathcal{D}$ is a nonempty closed convex subset of $\mathcal{X}$. $\mathcal{D}$ is bounded and $x_p(., \psi) \in \mathcal{D}$ for all $\psi \in \mathcal{D}$. In fact, let $\psi \in \mathcal{D}$, then there exists a sequence $(\psi^k)_k \subset \{y_{np} : n \in \mathbb{N}\}$ such that

$$\psi^k = \sum_{i=1}^{n_k} \alpha_i^k y_{n_i^k p}, \alpha_i^k \geq 0, \sum_{i=1}^{n_k} \alpha_i^k = 1 \text{ and } \left\|\psi - \psi^k\right\|_{\mathcal{B}} \to 0 \text{ as } k \to \infty.$$

Therefore, we get

$$\begin{aligned}\|\psi\|_{\mathcal{B}} &\leq \left\|\psi^k\right\|_{\mathcal{B}} + \left\|\psi^k - \psi\right\|_{\mathcal{B}} \\ &\leq \sum_{i=1}^{n_k} \alpha_i^k \left\|y_{n_i^k p}\right\|_{\mathcal{B}} + \left\|\psi^k - \psi\right\|_{\mathcal{B}},\end{aligned}$$

and (5.12) yields $\|\psi\|_{\mathcal{B}} < N_1$. On the other hand, one has

$$\begin{aligned}x_p(.,\psi^k) &= \sum_{i=1}^{n_k} \alpha_i^k x_p(., y_{n_i^k p}) \\ &= \sum_{i=1}^{n_k} \alpha_i^k y_{n_i^k p+p},\end{aligned}$$

then $x_p(.,\psi^k) \in \mathcal{D}$. Since $\left\|x_p(.,\psi) - x_p(.,\psi^k)\right\|_{\mathcal{B}} \to 0$ as $k \to +\infty$, and $\mathcal{D}$ is closed we deduce that $x_p(.,\psi) \in \mathcal{D}$ for all $\psi \in \mathcal{D}$. Which implies that assumption (**H5.3**) is true and by Theorem 5.2.5, we conclude that Equation (5.11) has a p-periodic solution. ■

Corollary 5.3.2 *Assume that $\mathcal{B} = C_\gamma$, $\gamma > 0$. Under the conditions* (**H2.1**), (**H5.1**) *and* (**H5.4**), *Equation (5.11) has a mp-periodic solution, for a certain $m \in I\!N^*$.*

Proof. It was noted in Chapter 1 that if $\gamma > 0$, the phase space C_γ satisfies (**C**) and (**D2**). Moreover one has $K(\sigma) = 1$ and $M(\sigma) = e^{-\gamma\sigma}$. Hence

$$\begin{aligned}&\inf_{0<\sigma<p} \left\{ M(p-\sigma)[HK(\sigma) \sup_{0\leq t\leq\sigma} \|T_0(t)\| + M(\sigma)] \right\} \\ &= \inf_{0<\sigma<p} \left\{ H \sup_{0\leq t\leq\sigma} \|T_0(t)\| \, e^{-\gamma(p-\sigma)} \right\} + e^{-\gamma p}.\end{aligned}$$

From what we can see that for a certain $m \in I\!N^*$, (5.13) in Theorem 5.3.1 is satisfied with mp instead of p. Consequently, all conditions of Theorem 5.3.1 are verified and Equation (5.11) has a mp-periodic solution. ■

Next, we suppose that $\mathcal{B}$ is a normed uniform fading memory space, that means, $\mathcal{B}$ is normed and satisfies the extra axioms (**C**) and

(**D3**) $\|W_0(t)\| \to 0$ as $t \to +\infty$.

In the following theorem, we prove existence of a periodic solution without supposing condition (5.13).

Theorem 5.3.3 *Suppose that $\mathcal{B}$ is a normed uniform fading memory space. Under conditions* (**H2.1**), (**H5.1**) *and* (**H5.4**), *Equation (5.11) has a p-periodic solution.*

The proof is based on the following two results.

Lemma 5.3.4 *Let conditions* (**H2.1**) *and* (**H5.1**) *hold. Then, for any* $\varepsilon > 0$*, there exists a constant* $C_\varepsilon > 0$ *such that* $\alpha(\mathcal{U}_1(t)) \leq C_\varepsilon M(t-\varepsilon)$ *for* $t > \varepsilon$*.*

Proof. Let   be a bounded set in $\mathcal{X}$ such that $\alpha(\ ) = \beta$. For $\delta > 0$, there exists a finite cover $(\ _i)_{1\leq i\leq n}$ of   such that $diam(\ _i) < \beta + \delta$. Using Axiom $(\mathbf{A} - \mathbf{iii})$, we have for $t > \varepsilon$ and $\varphi, \phi \in\ $,

$$\begin{aligned} \|\mathcal{U}_1(t)\varphi - \mathcal{U}_1(t)\phi\|_{\mathcal{B}} \leq\ & K(t-\varepsilon) \sup_{\varepsilon\leq s\leq t} |T_0(s)\varphi(0) - T_0(s)\phi(0)| \\ & + M(t-\varepsilon) \|\mathcal{U}_1(\varepsilon)\varphi - \mathcal{U}_1(\varepsilon)\phi\|_{\mathcal{B}}. \end{aligned} \tag{5.14}$$

On the other hand

$$\|\mathcal{U}_1(\varepsilon)\varphi - \mathcal{U}_1(\varepsilon)\phi\|_{\mathcal{B}} \leq K(\varepsilon) \sup_{0\leq s\leq \varepsilon} |T_0(s)\varphi(0) - T_0(s)\phi(0)| + M(\varepsilon) \|\varphi - \phi\|_{\mathcal{B}},$$

which implies that

$$\|\mathcal{U}_1(\varepsilon)\varphi - \mathcal{U}_1(\varepsilon)\phi\|_{\mathcal{B}} \leq C_\varepsilon \|\varphi - \phi\|_{\mathcal{B}}, \tag{5.15}$$

where $C_\varepsilon = K(\varepsilon)H \sup_{0\leq s\leq\varepsilon} \|T_0(s)\| + M(\varepsilon)$. Consequently we have

$$\|\mathcal{U}_1(t)\varphi - \mathcal{U}_1(t)\phi\|_{\mathcal{B}} \leq K(t-\varepsilon) \sup_{\varepsilon\leq s\leq t} |T_0(s)\varphi(0) - T_0(s)\phi(0)| + M(t-\varepsilon)C_\varepsilon \|\varphi - \phi\|_{\mathcal{B}}. \tag{5.16}$$

  is bounded in $\mathcal{B}$ it follows by Axiom $(\mathbf{A}-(\mathbf{ii})')$ that $\{\psi(0) : \psi \in\ \}$ is bounded in E. Using the compactness of the semigroup $(T_0(t))_{t\geq 0}$, we can prove that $\{T_0(.)\psi(0) : \psi \in\ \}$ is relatively compact in $C([\varepsilon, t], E)$, then there exists a finite cover $(\Gamma_i)_{1\leq i\leq m}$ of $\{T_0(.)\psi(0) : \psi \in\ \}$ in $C([\varepsilon, t], E)$ such that $diam(\Gamma_i) < \delta$. Let $\ _{i,j} = \{\psi \in\ _i : T_0(.)\psi(0) \in \Gamma j\}$. Then $\ \subset \bigcup_{i,j}\ _{i,j}$. For $\varphi, \phi \in\ _{i,j}$ by (5.16) we have

$$\|\mathcal{U}_1(t)\varphi - \mathcal{U}_1(t)\phi\|_{\mathcal{B}} \leq K(t-\varepsilon)\delta + M(t-\varepsilon)C_\varepsilon(\beta+\delta) \|\varphi - \phi\|_{\mathcal{B}},$$

letting δ to zero we obtain

$$\alpha(\mathcal{U}_1(t)) \leq C_\varepsilon M(t-\varepsilon), \text{ for } t > \varepsilon.$$

This completes the proof of the lemma. ■

Theorem 5.3.5 *[33] Let* Z *be a Banach space and* $P : Z \to Z$ *a linear affine map, that is* $Px = Lx + y$ *where* L *is a bounded linear map and* $y \in Z$ *is fixed. If* $\mathrm{Im}(I - L)$ *is closed and there exists an* $x_0 \in Z$ *such that* $(P^n x_0)_{n\geq 0}$ *is bounded, then* P *has at least one fixed point.*

Proof of Theorem 5.3.3. Let $x(.,\varphi,f)$ be the solution of Equation (5.11). Then $x(.,\varphi,f) = x(.,\varphi,0)+x(.,0,f)$, where $x(.,\varphi,0)$ and $x(.,0,f)$ are, respectively, the solutions of Equation (5.11) with $f=0$ and $\varphi=0$. It follows that the Poincaré map P_p is given by $P_p\varphi = L\varphi+\psi$, with $L\varphi = x_p(.,\varphi,0)$ and $\psi = x_p(.,0,f)$. According to Proposition 5.2.6, L is decomposed as follows $L = \mathcal{U}_1(p)+\mathcal{U}_2(p)$, where $(\mathcal{U}_1(t))_{t\geq 0}$ is the solution semigroup of Equation (5.8) and $\mathcal{U}_2(p)$ is compact. By Proposition 1.1.2, $K(.)$ and $M(.)$ can be chosen such that $K(.)$ is bounded and $M(t)\to 0$ as $t\to+\infty$. Let $\varepsilon>0$, and $t_0>0$ such that $C_\varepsilon M(t_0-\varepsilon)<1$. Then, by Lemma 5.3.4, we have

$$\omega_e(\mathcal{U}_1) \leq \frac{\log\left(\alpha(\mathcal{U}_1(t_0))\right)}{t_0} < 0.$$

Hence $(3.9)_2$ gives

$$r_e(\mathcal{U}_1(p)) = e^{p\omega_e(\mathcal{U}_1)} < 1.$$

On the other hand, we have

$$L^n = \mathcal{U}_1^n(p)+V_n, \text{ for all } n>1,$$

where V_n is a compact operator. Using the formula (3.7) we get

$$r_e(L) = r_e(\mathcal{U}_1(p)) < 1.$$

Consequently we obtain that $\mathrm{Im}(I-L)$ is closed. Let $y(.,\phi,f)$ be the bounded solution. Then $\left(P_p^n\phi\right)_{n\geq 0} = (y_{np}(.,\phi,f))_{n\geq 0}$ is bounded. Consequently, by Theorem 5.3.5, we deduce that the map P_p has a fixed point in $\mathcal{B}$, which gives a p-periodic solution to Equation (5.11).
■

5.4 Application

In order to apply the abstract result of the previous section, we consider the following partial functional differential equation with infinite delay

$$\begin{cases} \dfrac{\partial}{\partial t}w(t,\xi) = \dfrac{\partial^2}{\partial \xi^2}w(t,\xi) + b(t)\displaystyle\int_{-\infty}^{0} G(\theta)w(t+\theta,\xi)d\theta + g(t,\xi), \\ \qquad\qquad\qquad\qquad t\geq 0,\ 0\leq\xi\leq\pi, \\ w(t,0) = w(t,\pi) = 0, \quad t\geq 0, \\ w(\theta,\xi) = w_0(\theta,\xi), \quad -\infty<\theta\leq 0, \quad 0\leq\xi\leq\pi, \end{cases} \tag{5.17}$$

where $b : I\!R^+ \to I\!R$ is a continuous p-periodic function, $g : I\!R^+ \times [0,\pi] \to I\!R$ is a continuous function, p-periodic with respect to time, G is a positive function on $(-\infty, 0]$ and $w_0 : (-\infty, 0] \times [0,\pi] \to I\!R$ is a continuous function.

Equation (5.17) can be written as the abstract equation (5.11). Indeed, we choose $E = C([0,\pi]; I\!R)$ endowed with the uniform norm topology and we consider the operator $A : D(A) \subset E \to E$ defined by

$$\begin{cases} D(A) = \left\{ y \in C^2([0,\pi] : I\!R) : y(0) = y(\pi) = 0 \right\}, \\ Ay = y''. \end{cases}$$

From [40], we know that

$$(0, +\infty) \subset \rho(A) \text{ and } \left| (\lambda I - A)^{-1} \right| \leq \frac{1}{\lambda}, \text{ for } \lambda > 0.$$

This implies that assumption (**H2.1**) is satisfied. Moreover one has

$$\overline{D(A)} = \{ y \in E : y(0) = y(\pi) = 0 \}.$$

It is known from [66] that the space C_γ, $\gamma > 0$, which is introduced in Lemma 5.2.8, is a normed uniform fading memory space.

We suppose that :

(i) $G(.)e^{-\gamma \cdot}$ is integrable on $(-\infty, 0]$,

(ii) $w_0 \in C((-\infty, 0] \times [0,\pi] : I\!R)$, with $\lim_{\theta \to -\infty} \left(e^{\gamma\theta} \sup_{0 \leq \xi \leq \pi} |w_0(\theta, \xi)| \right)$ exists, and $w_0(0,0) = w_0(0,\pi) = 0$.

Let

$$\begin{cases} L(t,\phi)(\xi) = b(t) \displaystyle\int_{-\infty}^{0} G(\theta)\phi(\theta)(\xi) d\theta, \quad t \geq 0,\ \xi \in [0,\pi],\ \phi \in C_\gamma, \\ f(t)(\xi) = g(t,\xi),\ t \geq 0,\ \xi \in [0,\pi]. \end{cases}$$

Then L is continuous from $I\!R^+ \times C_\gamma$ into E, linear with respect to the second argument and p-periodic in t, $f : I\!R^+ \to E$ is continuous p-periodic. If we put

$$\begin{cases} x(t)(\xi) = w(t,\xi), \quad t \geq 0,\ \xi \in [0,\pi], \\ \varphi(\theta)(\xi) = w_0(\theta,\xi), \quad \theta \leq 0,\ \xi \in [0,\pi]. \end{cases}.$$

Then Equation (5.17) is written in E as follows

$$\begin{cases} \dfrac{d}{dt} x(t) = Ax(t) + L(t, x_t) + f(t),\ t \geq 0, \\ x_0 = \varphi. \end{cases} \tag{5.18}$$

We can see that assumption (ii) implies that $\varphi \in C_\gamma$ and $\varphi(0) \in \overline{D(A)}$.

Let A_0 be the part of the operator A in $\overline{D(A)}$ given by

$$\begin{cases} D(A_0) = \left\{y \in C^2([0,\pi] : I\!R) : y(0) = y''(0) = y(\pi) = y''(\pi) = 0\right\}, \\ A_0 y = y''. \end{cases}$$

A_0 generates a compact C_0-semigroup $(T_0(t))_{t\geq 0}$ in $\overline{D(A)}$. Moreover $\|T_0(t)\| \leq e^{-t}$, $t \geq 0$.

Assume further that

(iii) there exists $d \in]0,1[$ such that $0 < l \int_{-\infty}^{0} G(\theta) d\theta < 1 - d$,

where $l = \sup_{0\leq t\leq p} |b(t)|$.

Let $\rho = 1 + \dfrac{m}{d}$, where $m = \sup_{0\leq t\leq p} |f(t)|$. Then we have

Lemma 5.4.1 *Under the above assumptions, let $\varphi \in C_\gamma$ such that $|\varphi(\theta)| < \rho$ for all $\theta \in (-\infty, 0]$. Then the integral solution of Equation (5.18) is bounded by ρ on $I\!R^+$.*

Proof. Let $x(.,\varphi)$ be the integral solution of Equation (5.18). Then $|x(t,\varphi)| < \rho$ for $t \geq 0$. We proceed by contradiction. Suppose that there exists $t_0 > 0$ such that $|x(t_0,\varphi)| > \rho$. Let

$$t_1 = \inf\{t > 0 : |x(t,\varphi)| > \rho\}.$$

By continuity, it follows that

$$|x(t_1,\varphi)| = \rho, \tag{5.19}$$

and there exists $\delta > 0$ such that $|x(t,\varphi)| > \rho$ for $t \in (t_0, t_0 + \delta)$. Since

$$x(t,\varphi) = T_0(t)\varphi(0) + \lim_{\lambda\to+\infty} \int_0^t T_0(t-s) B_\lambda (L(s, x_s(.,\varphi)) + f(s))\, ds, \text{ for } t \geq 0.$$

We have

$$|x(t_1,\varphi)| \leq e^{-t_1}\rho + \int_0^{t_1} e^{-(t_1-s)} |L(s, x_s(.,\varphi))|\, ds + \int_0^{t_1} e^{-(t_1-s)} m ds,$$

Since $-\infty < s + \theta \leq t_1 + \theta \leq t_1$ for all $\theta \in (-\infty, 0]$ and $s \in [0, t_1]$, we have

$$\begin{aligned} |L(s, x_s(.,\varphi))| &= |b(s)| \int_{-\infty}^{0} G(\theta) |x(s+\theta,\varphi)|\, d\theta \\ &\leq \rho l \int_{-\infty}^{0} G(\theta) d\theta. \end{aligned}$$

This gives by assumption (iii)

$$|x(t_1, \varphi)| \leq e^{-t_1}\rho + (\rho - \rho d + m)\left(1 - e^{-t_1}\right).$$

Therefore

$$\begin{aligned} |x(t_1, \varphi)| &\leq \rho - d\left(1 - e^{-t_1}\right) \\ &< \rho. \end{aligned}$$

Which contradicts (5.19) by definition of t_1. This completes the proof of the lemma. ■

Finally, all conditions of Theorem 5.3.3 are satisfied, and consequently, Equation (5.18) has a p-periodic solution.

CONCLUSION ET PERSPECTIVES

Dans ce travail, nous avons étudié des aspects quantitatif et qualitatif pour une classe d'équations différentielles à retard infini et de type neutre dans un espace de Banach. Nous nous sommes intéressés dans un premier lieu aux problèmes d'existence, d'unicité et de régularité des solutions. Puis, nous avons étudié des problèmes de stabilté et d'attractivité. Enfin, nous avous traité l'existence de solutions périodiques.

An terme de ce travail, nous avons pensé à un certain nombre de problèmes ouverts. Dans la suite nous donnons quelques questions que nous projetons étudier dans le futur.

Dans le chapitre 3, nous avons étudié l'existence de solutions intégrales dans le cas où F est continue et localement Lipschitzienne par rapport à la deuxième variable. Nous avons l'intention de traiter l'existence sous les conditions de Carathéodory: F est mesurable par rapport à la première variable et continue par rapport à la deuxième variable.

Nous avons aussi formulé un critère de stabilité dans le cas où la part de A dans $\overline{D(A)}$ engendre un C_0 semi-groupe compact $(T_0(t))_{t\geq 0}$. Nous projetons généraliser le résultat obtenu dans le cas où $(T_0(t))_{t\geq 0}$ n'est pas compact mais uniformément continu, à savoir $\lim_{t\to t_0} \|T_0(t) - T_0(t_0)\|_{\mathcal{L}(\overline{D(A)})} = 0$, pour tout $t_0 > 0$.
Nous avons aussi donné des conditions suffisantes pour l'existence d'un attracteur global pour une classe d'équations différentielles à retard infini dissipatives. Nous comptons exploiter cette propriété pour étudier l'existence et la stabilté des solutions périodiques.

Nous avons étudié, dans le chapitre 4, l'existence de solutions intégrales et strictes pour une classe d'équations différentielles de type neutre à retard infini. Dans le cas autonome, nous avons démontré que la solution définit un semi-groupe $(U(t))_{t\geq 0}$ sur l'espace de phase $\mathcal{B}$. Auquel cas, nous avons formulé un critère de stabilité. Par la suite, la question que nous nous posons est la suivante: peut on décomposer le semi groupe $(U(t))_{t\geq 0}$ sous la forme suivante : $U(t) = V(t) + W(t)$, $t \geq 0$ où $(V(t))_{t\geq 0}$ est un C_0 semi-groupe exponentiellement stable et $W(t)$ est un opérateur compact pour $t > 0$? Une réponse positive à cette question nous permettra d'examiner la stabilité du semi groupe solution dans le cas linéaire par une équation caractéristique. Dans le cas non linéaire, nous étudierons l'existence d'un attracteur global. Un autre volet auquel nous nous intéressons, est celui de l'existence de solutions bornés et périodiques.

Dans le chapitre 5, nous avons étudié l'existence de solutions périodiques dans le cas où F est périodique en t. Nous nous sommes intéressés par l'existence de solutions presque-périodiques dans le cas où F est presque-périodique en t uniformément par rapport à la deuxième variable. Nous avons l'intention de traiter une situation intermédiaire, à savoir, le cas où F est quasi-périodique, c'est à dire, $F = F_1 + F_2$, avec F_1 et F_2 sont respectivement, p_1-périodique et p_2-périodique en t et satisfaisant la condition de noncommensurabilité, c'est à dire $\frac{p_1}{p_2} \notin \mathbb{Q}$.

Bibliography

[1] M. Adimy, Abstract semilinear functional differential equations with non-dense domain, *Publications internes de l'Université de Pau et des Pays de L'Adour, URA 1204 Pau 95/18, (1995).*

[2] M. Adimy, Integrated semigroups and delay differential equations, *J. Math. Anal. Appl., Vol. 177, N. 1, 125-134, (1993).*

[3] M. Adimy, Semi-groupes intégrés et équations aux dérivées partielles non locales en temps, *Thèse d'Habilitation, Université de Pau et des Pays de L'Adour, (1999).*

[4] M. Adimy, H. Bouzahir and K. Ezzinbi, Existence for a class of partial functional differential equations with infinite delay, *Nonlinear Analysis, Theory, Methods and Applications, to appear, (2001).*

[5] M. Adimy, H. Bouzahir and K. Ezzinbi, Local existence and stability for a class of partial functional differential equations with infinite delay, *Nonlinear Analysis, Theory, Methods and Applications, to appear, (2001).*

[6] M. Adimy, H. Bouzahir and K. Ezzinbi, Existence and stability for some neutral functional differential equations with infinite delay, *(in progress).*

[7] M. Adimy and K. Ezzinbi, Local existence and linearized stability for partial functional differential equations, *Dynamic Systems and Applications, Vol. 7, N. 3, 389-404, (1998).*

[8] M. Adimy et K. Ezzinbi, Equations de type neutre et semi-groupes intégrés, *C. R. Acad. Sci. Paris, t. 318, Sér. I, 529-534, (1994).*

[9] M. Adimy et K. Ezzinbi, Semi-groupes intégrés et équations différentielles à retard en dimension infinie, *C. R. Acad. Sci. Paris, t. 323, Sér. I, 481-486, (1996).*

[10] M. Adimy and K. Ezzinbi, A class of linear partial neutral functional differential equations with nondense domain, *J. Differential Equations, Vol. 147, N. 2, 285-332, (1998).*

[11] M. Adimy and K. Ezzinbi, Strict solutions of nonlinear hyperbolic neutral differential equations, *Applied Mathematics Letters., Vol. 12, N. 1, 107-112, (1999).*

[12] M. Adimy and K. Ezzinbi, Existence and linearized stability for partial neutral functional differential equations with nondense domains, *Differential Equations and Dynamical Systems, Vol. 7, 371-417, (1999).*

[13] M. Adimy, K. Ezzinbi and M. Laklach, Existence of solutions for a class of partial neutral differential equations, *C. R. Acad. Sci. Paris Sér. I Math. Vol. 330, N. 11, 957–962, (2000).*

[14] M. Ait Babram, R. Benkhalti and K. Ezzinbi, Periodic solutions of functional differential equations of neutral type, *J. Math. Anal. Appl., Vol. 204, N. 3, 898-909, (1996).*

[15] E. Ait Dads, Contribution à l'existence de solutions pseudo presque périodiques d'une équation fonctionnelle non linéaire, *Thèse d'Etat, Faculté des sciences Semlalia, Université Cadi Ayyad, Marrakech, (1994).*

[16] E. Ait Dads and K. Ezzinbi, Existence of positive pseudo-almost-periodic solutions for some nonlinear infinite delay integral equations arising in epidemic problems, *Nonlinear Analysis, Theory, Methods and Applications, Vol. 41, N. 1-2, 1-13, (2000).*

[17] E. Ait Dads and K. Ezzinbi, Pseudo almost periodic solutions for some delay differential equations, *J. Math. Anal. Appl., Vol. 201, N. 3, 840-850, (1996).*

[18] W. Arendt, Resolvent positive operators and integrated semigroups, *Proc. London Math. Soc. (3), Vol. 54, N. 2, 321-349, (1987).*

[19] W. Arendt, Vector valued Laplace transforms and Cauchy problems, *Israelien J. Math., Vol. 59, N. 3, 327-352, (1987).*

[20] O. Arino, A. Burton and J. R. Haddock, Periodic solutions to functional differential equations, *Proc. Royal Soc. Edinburgh A, Vol. 101, N. 3-4, 253-271, (1985).*

[21] O. Arino and My. L. Hbid, Existence of periodic solutions for a delay differential equation via the Poincaré procedure, *J. Differential Equations and Dynamical Systems, Vol. 4, N. 2, 125-148, (1996).*

[22] O. Arino, My. L. Hbid and R. Bravo de la Parra, A mathematical model of growth of population of fish in the larval stage: density and dependence effects, *Math. Biosc., Vol. 150, N. 1, 1-20, (1998).*

[23] O. Arino and E. Sanchez, Linear theory of abstract functional differential equations of retarded type, *J. Math. Anal. Appl., Vol. 191, N. 3, 547-571, (1995).*

[24] O. Arino and E. Sanchez, A variation of constants formula for an abstract functional differential equations of retarded type, *Differential and Integral Equations, Vol. 6, N. 9, 1305-1320, (1996).*

[25] D. Bainov and P. Simeonov, Integral Inequalities and Applications, *Kluwer Academic Publishers, (1992).*

[26] R. Benkhalti, H. Bouzahir and K. Ezzinbi, Existence of a periodic solution for some partial functional differential equations with infinite delay, *J. Math. Anal. Appl., to appear, (2001).*

[27] L. L. Bonilla and A. Linan, Relaxation oscillations, pulses, and travelling waves in the diffusive Volterra delay-differential equations, *SIAM J. Appl. Math., Vol. 44, N. 2, 369-391, (1984).*

[28] H. Bouzahir and K. Ezzinbi, Global attractor for a class of partial functional differential equations with infinite delay, *in "Topics in Functional Differential and Difference Equations", eds. P. Freitas and T. Faria, Fields Inst. Commun., Vol. 29, Amer. Math. Soc., Providence, RI, in press (2001).*

[29] H. Brezis, Operateurs Maximaux Monotones, *North-Holland, Amsterdam, (1973).*

[30] N. F. Britton, Spatial structures and periodic travelling waves in an integro-differential reaction diffusion population model, *SIAM J. Appl. Math., Vol. 50, N. 6, 1663-1688, (1990).*

[31] T. A. Burton, Stability and Periodic Solutions of Ordinary and Functional Differential Equations, *Academic Press, Orlando, (1985).*

[32] S. Busenberg and B. Wu, Convergence theorems for integrated semigroups, *J. Differential and Integral Equations, Vol. 5, N. 3, 509-520, (1992).*

[33] S. N. Chow and J. K. Hale, Strongly limit-compact maps, *Funkcialaj Ekvaciaoj, Vol. 17, 31-38, (1974).*

[34] D. S. Cohen, P. S. Hagan and H. C. Simpson, Spacial structures in predator-prey communities with hereditary effects and diffusion, *Math Biosci. Vol 44, 167-177, (1979).*

[35] B. C. Coleman and V. J. Mizel, Norms and semigroups in the theory of fading memory, *Arch. Rational Mech. Anal., Vol. 23, 87-123, (1966).*

[36] B. C. Coleman and V. J. Mizel, On the general theory of fading memory, *Arch. Rational Mech. Anal., Vol. 29, 18-31, (1968).*

[37] B. C. Coleman and V. J. Mizel, On the stability of functional differential equations, *Arch. Rational Mech. Anal., Vol. 30, 173-196, (1968).*

[38] C. Corduneanu and V. Lakshmikantham, Equations with unbounded delay: a survey, *Nonlinear Analysis, Theory, Methods and Applications, Vol. 4, N. 5, 831-877, (1980).*

[39] G. Da Prato and A. Lunardi, Solvability on the real line of a class of linear Volterra integro-differential equations of parabolic type, *Annali Mat. Pura Appl. (4), Vol. 150, 67-117, (1988).*

[40] G. Da Prato and E. Sinestrari, Differential operators with nondense domains, *Ann. Scuola Norm. Sup. Pisa Cl. Sci., Vol. 14, N. 2, 285-344, (1987).*

[41] W. Desch and W. Schappacher, Linearized stability for nonlinear semigroups, *in "Differential Equations in Banach Spaces" eds. A. Favini and E. Obrecht, Lecture Notes in Math., Springer-Verlag, Vol. 1223, 61-73, (1986).*

[42] O. El Mennaoui, Traces des semi-groupes holomorphes singuliers à l'origine et comportement asymptotique, *Thèse de l'Université de Franche Comté, N. 291, (1992).*

[43] K. Engel and R. Nagel, One Parameter Semigroups for Linear Evolution Equations, *Graduate Texts in Math., Vol. 194, Springer Verlag, (2000).*

[44] K. Ezzinbi, Contribution à l'étude des équations différentielles en dimension fine et infinie : existence et aspects qualitatifs; application à des problèmes de dynamique de populations, *Thèse d'état, Faculté des sciences Semlalia, Université Cadi Ayyad de Marrakech, N. d'ordre 167, (1997).*

[45] K. Ezzinbi and H. Tamou, Abstract functional differential equations, *Dynam. Discrete Continuous and Impulsive Systems, to appear, (2001).*

[46] A. Grabosch and U. Moustakas, A semigroup approach to retarded differential equations, *in "One-parameter Semigroups of Positive Operators", Lecture Notes in Mathematics, ed. R. Nagel, Vol. 1184, Springer-Verlag, Berlin, New-York, 219-232, (1986).*

[47] G. Guhring, F. Raliger and W. M. Ruess, Linearized stability for semilinear non-autonomuous evolution equations with applications to retarded differential equations, *Differential and Integral Equations, Vol. 13, N. 4-6, April-June, 503-527, (2000).*

[48] J. Haddock, S. Ruan, J. Wu and H. Xia, Comparison theorems of Liapunov-Razumikhan type for NFDEs with infinite delay, *Can. J. Math., Vol. 47, N. 3, 500-526, (1995).*

[49] J. Haddock and J. Terjeki, On the location of positive limit sets for autonomous functional differential equations with infinite delay, *J. Differential Equations, Vol. 86, N. 1, 1-32, (1990).*

[50] J. K. Hale, Retarded equations with infinite delays, *in "Functional Differential Equations and Approximation of Fixed Points", Proceedings, Bonn, Lecture Notes in Mathematics, Springer-Verlag, Berlin/ Heidelberg/ New York, Vol. 730, 157-193, (1978).*

[51] J. K. Hale, Asymptotic Behavior of Dissipative Systems, *Mathematical Surveys and Monographs, N. 25, American Mathematical Society, Providence, RI, (1988).*

[52] J. K. Hale, Dynamical systems and stability, *J. Math. Anal. Appl., Vol. 26, 35-59, (1969).*

[53] J. K. Hale, Partial neutral functional differential equations, *Rev. Roumaine Math. Pure Appl., Vol. 39, N. 4, 339-344, (1994).*

[54] J. K. Hale, Coupled oscillators on a circle, *Dynamical phase transitions (São Paulo, 1994), Resenhas, Vol. 1, N. 4, 441-457, (1994).*

[55] J. K. Hale, and J. Kato, Phase space for retarded equations with infinite delay, *Funkcial. Ekvac., Vol. 21, N. 1, 11-41, (1978).*

[56] J. K. Hale and S. Verduyn-Lunel, Introduction to Functional Differential Equations, *Applied Mathematical Sciences, Vol. 99, Springer-Verlag, New York, (1993).*

[57] My. L. Hbid, Contribution à l'étude des perturbations des équations différentielles à retard, *Thèse d'état, Faculté des sciences, Université Mohamed V, Rabat, (1993).*

[58] H. R. Henriquez, Periodic solutions of quasi-linear partial functional differential equations with unbounded delay, *Funkcialaj Ekvaciaoj, Vol. 37, N. 2, 329-343, (1994).*

[59] H. R. Henriquez, Approximation of abstract functional differential equations with unbounded delay, *Indian J. Pure Appl. Math., Vol. 27, N. 4, 357-386, (1996).*

[60] H. R. Henriquez, Regularity of solutions of abstract retarded functional differential equations with unbounded delay, *Nonlinear Analysis, Theory, Methods and Applications, Vol. 28, N. 3, 513-531, (1997).*

[61] E. Hernandez and H. R. Henriquez, Existence results for partial neutral functional differential equations with unbounded delay, *J. Math. Anal. Appl., Vol. 221, N. 2, 452-475, (1998).*

[62] E. Hernandez and H. R. Henriquez, Existence of periodic solutions for partial neutral functional differential equations with unbounded delay, *J. Math. Anal. Appl., Vol. 221, N. 2, 499-522, (1998).*

[63] E. Hernandez, Regular solutions for partial neutral functional differential equations with unbounded delay, (*preprint).*

[64] M. Hieber, Integrated semigroups and differential operators on L^p, *Dissertation, (1989).*

[65] Y. Hino and S. Murakami, A generation of processes and stabilities in abstract functional differential equations, *Funkcialaj Ekvacioj, Vol. 41, N. 2, 235-255, (1997).*

[66] Y. Hino, S. Murakami and T. Naito, Functional Differential Equations with Infinite Delay, *Lecture Notes in Mathematics, Springer-Verlag, Berlin, Vol. 1473, (1991).*

[67] Y. Hino, S. Murakami and T. Yoshizawa, Existence of almost periodic solutions of some functional differential equations with infinite delay in a Banach space, *Tôhoku Math. J. (2), Vol. 49, N. 1, 133-147, (1997).*

[68] F. Kappel and W. Schappacher, Some considerations to the fundamental theory of infinite delay equations, *J. Differential Equations, Vol. 37, N. 2, 141-183, (1980).*

[69] A. G. Kartsatos and M. Parrott, The weak solution of a functional differential equation in a general Banach space, *J. Differential Equations, Vol. 75, N. 2, 290-302, (1988).*

[70] S. Kato, Almost periodic solutions of functional differential equations with infinite delays in a Banach space, *Hokkaido Math. J., Vol. 23, N. 3, 465-474, (1994).*

[71] H. Kellermann and M. Hieber, Integrated semigroups, *J. Fun. Anal., Vol. 15, N. 1, 160-180, (1989).*

[72] K. Kunisch and W. Schappacher, Variation of constants formula for partial differential equations with delay, *Nonlinear Analysis, Theory, Methods and Applications, Vol. 5, N. 2, 123-142, (1981).*

[73] K. Kunisch and W. Schappacher, Necessary conditions for partial differential equations with delay to generate a C_0-semigroup, *J. Differential Equations, Vol. 50, N. 1, 49-79, (1983).*

[74] V. Lakshmikantham, L. Wen and B. Zhang, Theory of Differential Equations with Unbounded Delay, *Mathematics and its Applications, Vol. 298, Kluwer Academic Publishers Group, Dordrecht, (1994).*

[75] J. Liang and T. Xiao, Functional differential equations with infinite delay in Banach spaces, *Int. J. Math. Sci., Vol. 14, N. 3, 497-508, (1991).*

[76] X. Lin, J. W.-H. So and J. Wu, Centre manifolds for partial functional differential equations with delays, *Proceedings of the Royal Society of Edinburgh, Sect. A 122, N. 3-4, 237-254, (1992).*

[77] X. Liu and D. Y. Xu, Uniform asymptotic stability of abstract differential equations, *J. Math. Anal. Appl., Vol. 216, N. 2, 626-643, (1997).*

[78] S. M. Lenhart and C. C. Travis, Stability of functional differential equations, *J. Differential Equations, Vol. 58, N. 2, 212-227, (1985).*

[79] L. Maniar and A. Rhandi, Extrapolation and inhomogeneous retarded differential equations on infinite dimensional spaces, *Rend. Circ. Mat. Palermo (2), Vol. 47, N. 2, 331-346, (1998).*

[80] M. Memory, Stable and unstable manifolds for partial functional differential equations, *Nonlinear Analysis, Theory, Methods and Applications, Vol. 16, N. 2, 131-142, (1991).*

[81] M. Memory, Invariant manifolds for partial functional differential equations, *in Mathematical Population Dynamics, eds. O. Arino, D. E. Axelrod and M. Kimmel), Lecture Notes in Pure and Appl. Math., Vol. 131, Dekker, New York, 223-232, (1991).*

[82] J. Milota, Stability and saddle point property for a linear autonomous functional parabolic equations, *Comment. Math Univ. Carolia., Vol. 27, N. 1, 87-101, (1986).*

[83] J. Milota, Stability in models with long memories, *in Biomathematics and Related Computational Problems, ed. L. M. Ricciardi, Kluwer Academic Publishers, 523-527, (1988).*

[84] S. Murakami, Stability for functional differential equations with infinite delay in Banach space, (*unpublished).*

[85] S. Murakami, Stable equilibrium point of some diffusive functional differential equations, *Nonlinear Analysis, Theory, Methods and Applications, Vol. 25, N. 9-10, 1037-1043, (1995).*

[86] S. Murakami and T. Naito, Some properties of phase spaces for functional differential equations with infinite delay, *in Differential Equations, Lecture Notes in Pure and Applied Mathematics, Vol. 118, Marcel Dekker, Inc., 507-514, (1989).*

[87] S. Murakami and T. Yoshizawa, Relationships between BC-stabilities and ρ-stabilities in functional differential equations with infinite delay, *Tohoku Math. J. (2), Vol. 44, N. 1, 45-57, (1992).*

[88] S. Murakami and T. Yoshizawa, Asymptotic behavior in a system of Volterra integrodifferential equations with diffusion, *Dynam. Systems Appl., Vol. 3, N. 2, 175-188, (1994).*

[89] R. Nagel (ed.), One-parameter Semigroups of Positive Operators, *Lecture Notes in Mathematics, Vol. 1184, Springer-Verlag, Berlin/New-York, (1986).*

[90] R. Nagel and E. Sinestrari, Nonlinear hyperbolic Volterra integrodifferential equations, *Nonlinear Analysis, Theory, Methods and Applications, Vol. 27, N. 2, 167-186, (1996).*

[91] T. Naito, J. S. Shin and S. Murakami, On solution semigroups of general functional differential equations, *Nonlinear Analysis, Theory, Methods and Applications, Vol. 30, N. 7, 4565-4576, (1997).*

[92] T. Naito, J. S. Shin and S. Murakami, The generator of the solution semigroup for the general linear functional-differential equation, *Bull. Univ. Electro-Comm., Vol. 11, N. 1, 29–38, (1998).*

[93] F. Neubrander, Integrated semigroups and their applications to the abstract Cauchy problems, *Pacific J. Math., Vol. 135, N. 1, 111-155, (1988).*

[94] R. D. Nussbaum, The radius of essential spectrum, *Duke Math. J., Vol. 37, 473-478, (1970).*

[95] M. E. Parrott, Positivity and a principle of linearized stability for delay differential equations, *Differential Integral Equations, Vol. 2, N. 2, 170-182, (1989).*

[96] M. E. Parrott, Linearized stability and irreducibility for a functional differential equation, *SIAM J. Math. Anal., Vol. 23, N. 3, 649-661, (1992).*

[97] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential equations, *Applied Mathematical Sciences, Vol. 44, Springer-Verlag, New York-Berlin, (1983).*

[98] H. Petzeltova, An existence theorem for semilinear functional parabolic equations, *Casopis pro pestovani matematiky, roc., Vol. 110, N. 3, 274-288, (1985).*

[99] H. Petzeltova, Solution semigroup and invariant manifolds for functional differential equations with infinite delay, *Mathematica Bohemica, Vol. 118, N. 2, 175-193, (1993).*

[100] H. Petzeltova, The Hopf bifurcation theorem for parabolic equations with infinite delay, *Mathematica Bohemica, Vol. 116, N. 2, 181-190, (1993).*

[101] H. Petzeltova and J. Milota, Resolvent operator for abstract functional differential equations with infinite delay, *Numer. Funct. Anal and Optimiz., Vol. 9, N. 7-8, 779-807, (1987).*

[102] A. Rhandi, Extrapolation methods to solve non-autonomous retarded partial functional differential equations, *Studia Math., Vol. 126, N. 3, 219-233, (1997).*

[103] S. G. Ruan and J. Wu, Reaction-diffusion equations with infinite delay, *Canad. Appl. Math. Quart. Vol. 2, N. 4, 485-550, (1994).*

[104] W. M. Ruess, Existence of solutions to partial evolution equations with infinite delay, *in Functional Analysis, eds. S. Dierof, S. Dineen and P. Domanski, de Gruyter, Berlin, 377-387, (1996).*

[105] W. M. Ruess, Existence and stability of solutions to partial functional differential equations with delay, *Advances in Differential Equations, Vol. 4, N. 6, 843-876, (1999).*

[106] W. M. Ruess, Existence of solutions to partial functional differential equations with delay, *in "Theory and Applications of Nonlinear Operators of Accretive and Monotone Type" ed., Athanassios G. Kartsatos, Lecture Notes Pure Appl. Math., Marcel Dekker, Vol. 178, 259-288, (1996).*

[107] W. M. Ruess and W. H. Summers, Linearized stability for abstract differential equations with delay, *J. Math. Anal. Appl., Vol. 198, N. 2, 310-336, (1996).*

[108] W. M. Ruess and W. H. Summers, Operator semigroups for functional differential equations with delay, *Transactions of the AMS, Vol. 341, N. 2, 695-719, (1996).*

[109] W. M. Ruess, W. H. Summers and H. William, Almost periodicity and stability for solutions to functional differential equations with infinite delay, *Differential Integral Equations, Vol. 9, N. 6, 1225-1252, (1996).*

[110] B. N. Sadovskii, On a fixed point principle, *Funct. Anal. Appl., Vol. 1, N. 2, 74-76, (1967).*

[111] K. Schumacher, Existence and continuous dependence for differential equations with unbounded delay, *Arch. Rational Mech. Anal., Vol. 64, N. 4, 315-335, (1978).*

[112] J. S. Shin, An existence theorem of a functional differential equation, *Funkcial. Ekvac., Vol. 30, 19-29, (1987).*

[113] J. S. Shin, Comparison theorems and uniqueness of mild solutions to semilinear functional differential equations in Banach spaces, *Nonlinear Analysis, Theory, Methods and Applications, Vol. 23, N. 7, 825-847, (1994).*

[114] J. S. Shin and T. Naito, Semi-Fredholm operators and periodic solutions for linear functional differential equations, *J. Differential Equations, Vol. 153, N. 2, 407-441, (1999).*

[115] J. S. Shin, T. Naito and N. V. Minh, On stability of solutions in linear autonomous functional differential equations, *Funkcialaj Ekvacioj, Vol. 43, 323-337, (2000).*

[116] H. Thieme, Integrated semigroups and integrated solutions to abstract Cauchy problems, *J. Math. Anal. Appl., Vol. 152, N. 2, 416-447, (1990).*

[117] H. Thieme, Semiflows generated by Lipschitz perturbations of non-densely defined operators, *Differential Integral Equations, Vol. 3, N. 6, 1035-1066, (1990).*

[118] C. C. Travis and G. F. Webb, Existence and stability for partial functional differential equations, *Trans. Amer. Math. Soc., Vol. 200, 395-418, (1974).*

[119] C. C. Travis and G. F. Webb, Existence, stability and compactness in the α-norm for partial functional differential equations, *Trans. Amer. Math. Soc., Vol. 240, 129-143, (1978).*

[120] G. F. Webb, Autonomous nonlinear functional differential equations and nonlinear semigroups, *J. Math. Anal. Appl., Vol. 46, 1-12, (1974).*

[121] G. F. Webb, Asymptotic stability for abstract nonlinear functional differential equations, *Proc. Amer. Math. Soc., Vol. 54, 225-230, (1976).*

[122] G. F. Webb, Theory of Nonlinear Age-Dependent Population Dynamics, *Monographs and Text books in Pure and Applied Mathematics, Vol. 89, Marcel Dekker, Inc., New York, (1985).*

[123] G. F. WEBB, Regularity of solutions to an abstract inhomogeneous linear differential equation, *Proc. Amer. Math. Soc., Vol. 62, N. 2, 271-277, (1977).*

[124] J. WU, Theory and Applications of Partial Functional Differential Equations, *Applied Mathematical Sciences, Vol. 119, Springer-Verlag, New York, (1996).*

[125] J. WU AND H. XIA, Self-sustained oscillations in a ring array of coupled lossless transmission lines, *J. Differential Equations, Vol. 124, N. 1, 247-278, (1996).*

[126] J. WU AND H. XIA, Rotating waves in neutral partial functional differential equations, *J. Dynamics and Differential Equations, Vol. 11, N. 2, 209–238, (1999).*

“Ce à quoi l'un s'était failli, l'autre y est arrivé et ce qui était inconnu à un siècle, le siècle suivant l'a éclairci, et les sciences et les arts ne se jettent pas en moule mais se forment et figurent peu à peu, en les maniant et polissant à plusieurs fois [...] : ce que ma force ne peut découvrir, je ne laisse pas de le sonder et essayer: et en retastant et pétrissant cette nouvelle matière, la remuant et l'echaufant, j'ouvre à celui qui me suit quelque facilité et la lui rends plus souple et plus maniable. Autant en fera le second au tiers qui est cause que la difficulté ne me doit pas désepérer, ni aussi peu mon impuissance...”

Montaigne, Les essaies, Livre II, Chapitre XII.

Louange a DIEU qui m'a aidé à atteindre ce résultat tant espéré